

# Proofs of Assertions in the Investigation of the Regular Polytope

Rod Thompson

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## Summary

The Regular Polytope has been shown to be a promising candidate for the rigorous representation of geometric objects, in a form that is computable using the finite arithmetic available on digital computers. It is also apparent that the approach can be used in the formulation of repeatable and verifiable definitions of geometric objects with the aim of defining robust information interchange protocols.

In the definition and investigation of the properties of the Regular Polytope representation, several propositions have been made, and the proofs developed. These proofs can be quite long and complex, and not suitable for presentation at conferences or publication in journals. They are gathered here to make them available for scrutiny in conjunction with the various papers and publications that refer to them.

The major assertions that this document addresses are:

- The set of Regular Polytopes forms and spans a Topological Space.
- There is a useful correspondence between the definition of equality of Regular Polytopes and the natural “point set” definition of equality..
- A simplified “programming shortcut” can be used to reduce significantly the algorithmic complexity of implementation.

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# 1 Introduction and Definitions

A topological space may be defined on a spatial primitive based on the union of a set of convex polyhedra, which in turn may be constructed as the intersection of half spaces defined by parametric equations with integer coefficients (within fixed size domain). This construction is referred to as a Regular Polytope, and has been researched to resolve issues of the mismatch between the mathematics (theory) and its implementation in a finite digital computer (Thompson 2005). This approach is closely related to the Spatial Constraint database approach (Kanellakis and Golding 1994) (based on  $\mathbf{I}^2$  or  $\mathbf{I}^3$ ), and shares with that approach the fact that it defines a topological space.

The definition of "Regular Polytope" can be found in Thompson (2005), but a loose definition is given here. In the following discussion, an operator in a circle is used to represent the result of executing that operation digitally, so that for example, the symbols  $\oplus \otimes \oslash \ominus$  are used to indicate computation of the sum, product, division and difference, while  $+ \cdot / -$  are used to indicate the actual sum, product quotient and difference of the real numbers or integers that the values represent – thus the statement:  $A\oplus B = A+B$  should be interpreted as an assertion that the computer addition of the variables gives the correct result. In general, Capital (non italic) letters (such as X) will be used for to represent computational integers, or the integer values they represent. Lower case non italic letters (such as x) will usually be used for the representation of domain-restricted rational numbers, but occasionally lower-case italic letters will be used for integer values (e.g.  $i=1..n$ ).

## 1.1 Half Space

A half space  $H(A,B,C,D)$  is defined as the set of all points  $P(X,Y,Z)$ ,  $-M \leq X,Y,Z \leq M$  where<sup>1</sup>:

$$\begin{aligned} & (A\otimes X \oplus B\otimes Y \oplus C\otimes Z \oplus D) \oslash 0 \text{ or} \\ & [(A\otimes X \oplus B\otimes Y \oplus C\otimes Z \oplus D) \ominus 0 \text{ and } A\ominus 0] \text{ or} \\ & [(B\otimes Y \oplus C\otimes Z \oplus D) \ominus 0 \text{ and } A\ominus 0 \text{ and } B\ominus 0] \text{ or} \\ & [(C\otimes Z \oplus D) \ominus 0 \text{ and } A\ominus 0, B\ominus 0 \text{ and } C\ominus 0] \end{aligned}$$

Where M is range of integer values allowed for point representations.

The values of the integers A,B,C and D, define the half space. In 3D applications, we place the restriction that  $-M < A,B,C < M$ ,  $-3M^2 < D < 3M^2$ , in 2D  $-M < A,B,C < M$ ,  $-3M^2 < D < 2M^2$ .  $H(0,0,0,0)$  is not a permitted half space.

Two special half spaces are defined,

$$\begin{aligned} H_0 &= H(0,0,0,-1) && \text{(i.e. the set of all points for which } -1 > 0). \\ H_\infty &= H(0,0,0,1) && \text{(i.e. the set of all points for which } 1 > 0). \end{aligned}$$

---

<sup>1</sup> In the earlier paper, an alternative definition using a Boolean parameter "S" was used, but this form is not used here.

## 1.2 Convex Polytope

A convex polytope is defined as the intersection of any finite number of half spaces. Convex polytope representation  $C$  is defined as:

$$C = \{H_i : H_i \in \mathbf{H}, i = 1..n\} \text{ where } \mathbf{H} \text{ is the set of all half spaces.}$$

Two special half spaces are defined,

$$C_0 = \{H_0\}. \quad (\text{i.e. a set containing only the empty half plane})$$

$$C_\infty = \{\} \quad (\text{i.e. the empty set – no constraints on allowed points}).$$

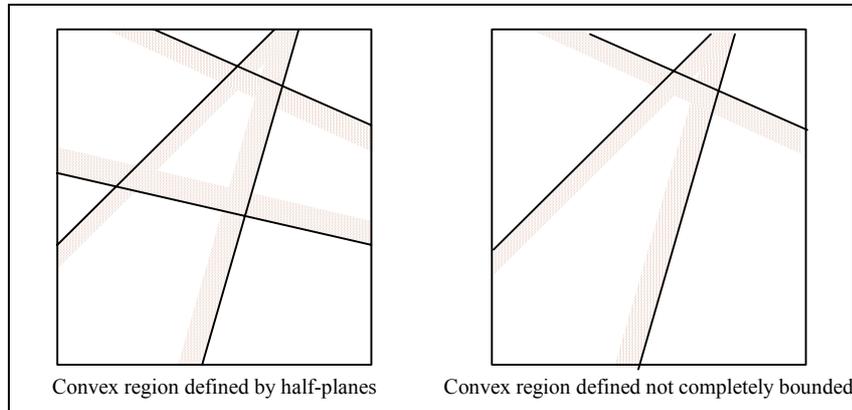


Figure 1 Convex polytopes defined by half planes (reproduced from (Thompson 2005)).

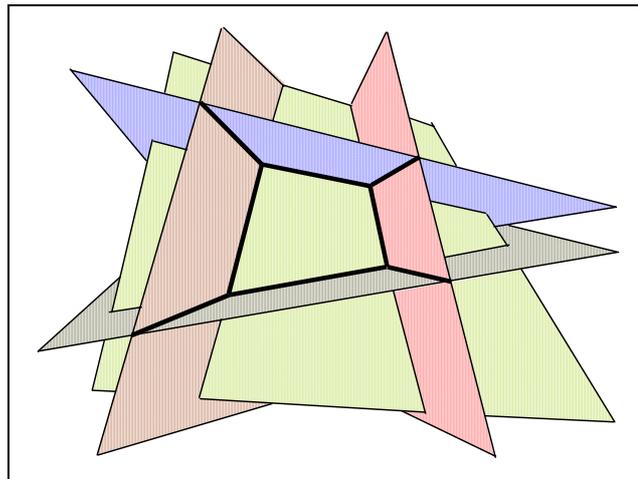


Figure 2 A convex polytope in 3D defined by half spaces

(reproduced from (Thompson 2005)).

The planes which represent the half space definitions should, of course, extend to “infinity”, but have been truncated to make the diagram clearer.

The point set interpretation of the definition of  $C$  is:

$$\text{Point } P \in C \Leftrightarrow P \in H_i, i = 1..n.$$

$$\text{Where } P = (X,Y,Z) \text{ and } (H_i = H(A_i B_i C_i D_i)).$$

This representation will be described by the shorthand  $C = \bigcap_{i=1,n} H_i$ .

### 1.3 Regular Polytope Representation

A polytope representation  $O$  is then defined as the union of a finite set of convex polytopes.

$$O = \{C_i : C_i \in \mathbf{C}, i=1,m\} \text{ where } \mathbf{C} \text{ is the set of all convex polytopes.}$$

Again, two special half spaces are defined,

$$O_{\Phi} = \{\}. \quad (\text{i.e. a set containing no convex polytopes})$$

$$O_{\infty} = \{C_{\infty}\} \quad (\text{i.e. a set containing the universal convex polytope}).$$

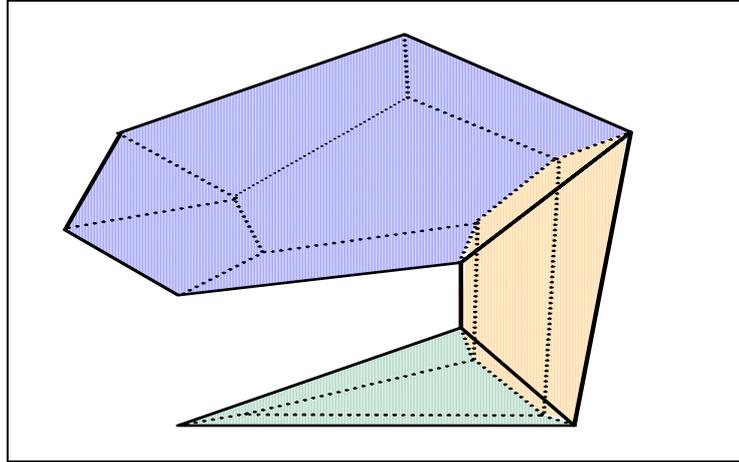


Figure 3 Definition of Regular Polytope from convex polytopes

The point set interpretation of the definition of  $O$  is:

$$\text{Point } P \in O \text{ iff } \exists C_i \in O : P \in C_i$$

The representation  $O$  will be described by the shorthand  $P = \bigcup_{i=1,m} C_i$ .

### 1.4 Integer Arithmetic

Define  $I$  as the set of all integer representations – thus  $I \in I$  is equivalent to the computer language coding of “int I”.

The following axioms are assumed for computational integer arithmetic:

Partial Closure:

$$\exists M_a : \forall I, J \in I; |I| \leq M_a; |J| \leq M_a \Rightarrow I \oplus J \in I$$

$$\exists M_m : \forall I, J \in I; |I| \leq M_m; |J| \leq M_m \Rightarrow I \otimes J \in I; |I \otimes J| \leq M_a$$

$$\forall I, J \in I: |I| \leq M_a; |J| \leq M_a; J \neq 0 \Rightarrow I \oslash J \in I.$$

If  $k=I/J$  (the real number obtained by dividing  $I$  by  $J$ ), then  $k-1 < I \oslash J \leq k$ .

It is assumed below that the integer arithmetic calculations give correct results – so that  $I \oplus J = I+J$ ,  $I \otimes J = IJ$ , within the range of closure of those operations. For this reason, the more usual mathematical operators such as +, - etc will be used below where there is no likelihood of confusion.

The values defined above,  $M_a$  (the additive maximum for integers),  $M_m$  (The multiplicative maximum for integers) depend on the implementation, but typically:

- $M_a$  is of the order of  $1*10^9$  for 4 byte integers,
- $M_m$  is of the order of 32767 for 4 byte integers
- $M_m$  is of the order of  $1*10^9$  for 8 byte integers

An integral point is an ordered triple of integers  $P = (X,Y,Z)$ , with  $X,Y,Z$  representing the Cartesian co-ordinate values. It is assumed that this introduces for integer representation an error

$$x - \frac{1}{2} \leq X \leq x + \frac{1}{2}, \quad y - \frac{1}{2} \leq Y \leq y + \frac{1}{2}, \quad z - \frac{1}{2} \leq Z \leq z + \frac{1}{2}.$$

#### 1.4.1 Domain-Restricted Rational Arithmetic

A domain restricted rational (dr\_rational) number  $r$  can be defined as an ordered pair of computational integers  $(I \text{ and } J, J > 0)^2$ , interpreted as having a value of  $I/J$ . The reason for the name “domain restricted rational” is that the values of  $I$  and  $J$  are constrained to a finite range of possible values (such as the numbers that can be represented as a 32 bit integer for example). The dr-rational numbers (unlike the rational numbers) do not form a field, and therefore cannot span a vector space. There are a number of other counter-intuitive properties of dr-rational numbers, for example that the sum or product of dr-rational numbers may not be dr-rational. A dr-rational point is an ordered triple of dr-rational numbers  $p = (x,y,z)$ , with  $x,y,z$  representing the Cartesian co-ordinate values.

The definition of the half space as a dr-rational point set is  $p(x,y,z) \in H(A,B,C,D)$  iff:

$$\begin{aligned} & (A \otimes x \oplus B \otimes y \oplus C \otimes z \oplus D) \ominus 0 \text{ or} \\ & [(A \otimes x \oplus B \otimes y \oplus C \otimes z \oplus D) \ominus 0 \text{ and } A \ominus 0] \text{ or} \\ & [(B \otimes y \oplus C \otimes z \oplus D) \ominus 0 \text{ and } A \ominus 0 \text{ and } B \ominus 0] \text{ or} \\ & [(C \otimes z \oplus D) \ominus 0 \text{ and } A \ominus 0, B \ominus 0 \text{ and } C \ominus 0] \end{aligned}$$

#### 1.5 The Topological Space Based on Integer or Dr-rational Representation

It is clearly impossible for the computer representations (dr\_rational or integer) of geometric objects to define metric space, since the density requirement is violated. There exist distinct points which do not have any representable points between them – owing to the finite nature of the computer.

It is, however possible to define a topological space within the computer representation. The advantage of this is that the results of topological theorems can be applied to the computer representations themselves, rather than to the mathematical abstraction.

---

<sup>2</sup> In the interest of brevity, in the following discussion, the requirement that  $q > 0$  is not explicitly addressed in every case. It is assumed that in any operation that leads to a pseudo-rational number  $r = (I/J)$  with  $J < 0$  will be converted to a valid number  $(-I/-J)$ . Thus this requirement is loosened to  $J \neq 0$ .

### 1.5.1 Operations on Half Spaces

$\bar{H}$  , the inverse of  $H(A,B,C,D)$  is defined as  $H(-A,-B,-C,-D)$ .

From the definition, it can be seen that this defines a valid half space, and that:

If  $(X,Y) \in H(A,B,C,D)$ , then  $(X,Y) \notin H(-A,-B,-C,-D)$ ; and

If  $(X,Y) \notin H(A,B,C,D)$ , then  $(X,Y) \in H(-A,-B,-C,-D)$ ;

so that this is a true inverse.

### 1.5.2 Operations on Convex Polytopes

For  $C_1 = \{H_{i1} : H_{i1}, i=1..n\}$ ,  $C_2 = \{H_{i2} : H_{i2}, i=1..m\}$ ,

Define  $C_1 \cap C_2 = \{H_{i1} : H_{i1}, i=1..n\} \cup \{H_{i2} : H_{i2}, i=1..m\}$ .

It is clear that this is a set of half spaces, and is therefore a convex polytope.

The symbology  $\bigcap_{j=1..l} C_j$  is taken to mean  $((((C_1 \cap C_2) \cap C_3) \dots C_l)$ .

That is to say, For  $C_j = \{H_{ij} : H_{ij}, i=1..n_i\}$ ,  $j=1..l$

$$\bigcap_{j=1..l} C_j = \{H_{ij} : i=1..n_p; j = 1..l\}$$

Note that  $C_1 \cup C_2$  is not in general a convex polytope, but does define a regular polytope.

### 1.5.3 Operations on Regular Polytopes

#### Union

For  $O_i = \{C_{ij} : C_{ij}, j=1..n_i\}$ ,  $i=1..m$

$$\text{Define } O = \bigcup_{i=1..m} O_i = \{C_{ij} : j=1..n_p; i = 1..n\}$$

Thus  $O$  is the union of a finite number of convex polytopes, and is therefore a regular polytope.

#### Intersection

For  $O_1 = \{C_{i1} : C_{i1}, i=1..n\}$ ,  $O_2 = \{C_{i2} : C_{i2}, i=1..m\}$

$$\text{Let } O_1 \cap O_2 = \{C_{i1} \cap C_{i2}, i=1..n, j=1..m\}$$

Since by definition, each  $C_{i1} \cap C_{i2}$  is itself a convex polytope, then the defined

$$O_1 \cap O_2 \text{ is a polytope.}$$

#### Containment

The set containment operator between two polytopes is defined as:

$$O_1 \subseteq O_2 \text{ if } \forall P : P \in O_1 \Rightarrow P \in O_2 \text{ for integral points}$$

$$\text{or } O_1 \subseteq O_2 \text{ if } \forall p : p \in O_1 \Rightarrow p \in O_2 \text{ for dr-rational points.}$$

(Note – there is another (stronger) set containment operator defined on polytopes – since a polytope is a set of convex polytopes  $O_1 \subset O_2$  could have been defined as

meaning that  $O_1$  consists of a subset of the convex polytopes that define  $O_2$ . This will be referred to if necessary as  $O_1 \subset_C O_2$ . Obviously  $O_1 \subset_C O_2 \Rightarrow O_1 \subset O_2$ .

It can be seen that  $\forall O : O \subseteq O_\infty$  since:

Let  $p = (x,y,z)$  be any point in  $O$

for any value of  $x,y,z$ , the value of  $(0 \otimes x \oplus 0 \otimes y \oplus 0 \otimes z \oplus 1)$  is 1, and therefore  $> 0$ .

Therefore  $p \in H_\infty$ ,

therefore  $p \in C_\infty$ ,

And finally  $p \in O_\infty$

By the reverse argument,  $\forall O : O_\phi \subseteq O$

### Inverse

The inverse (exterior) of a polytope is defined as follows:

The inverse of a convex set  $C = \bigcap_{i=1,n} H_i$ , is defined as:

$$\bar{C} = \bigcup_{i=1,n} \bar{H}_i .$$

This is the union of a finite set of convex polytopes, and is therefore a regular polytope representation by definition.

Finally, the inverse of a regular polytope  $O = \bigcup_{i=1,m} C_i$ , is defined as:

$$\bar{O} = \bigcap_{i=1,m} \bar{C}_i$$

This is the intersection of a finite number of regular polytopes, and so, by the above definition, is a regular polytope.

## 2 Set of Regular Polytopes forms a Topological Space

### 2.1 The Axioms of a Topological Space

Let  $\mathcal{O}$  be the set of all possible polytope representations.

To show that these polytopes form a topology, we need to confirm the axioms (Gaal 1964)

(O.1)  $O_\emptyset \in \mathcal{O}$  and  $O_\infty \in \mathcal{O}$

(O.2) if  $O_1 \in \mathcal{O}$  and  $O_2 \in \mathcal{O}$  then  $O_1 \cap O_2 \in \mathcal{O}$

(O.3) if  $O_i \in \mathcal{O}$  for all  $i \in I$  then  $\bigcup_{i \in I} O_i \in \mathcal{O}$

O.1.

Since the empty set is simply a special case of a polytope, this is obvious.  
By definition  $O_\infty$  is a polytope.

O.2

Defining  $O_1 \cap O_2$  as above (section 1.5.3), it is clear that  $O_1 \cap O_2 \in \mathcal{O}$ .

O.3

Defining  $O = \bigcup_{i=1..n} O_i$  as above (section 1.5.3), it is clear that  $\bigcup_{i=1..n} O_i \in \mathcal{O}$ .

The stronger requirement for a topological space, however is that  $\bigcup_{i \in I} O_i$  is an open set, where there is no requirement for the set  $I$  to be finite.

Consider  $\bigcup_{i \in I} O_i$ . Since the number of possible polytopes is finite, even if  $I$  represents an infinite set, there will be only a finite number of distinct  $O_i$ , such that  $O_i \neq O_j$  for  $i \neq j$ . Thus if  $O_i \in \mathcal{O}$  for all  $i \in I$  then  $\bigcup_{i \in I} O_i \in \mathcal{O}$ .

Note - the question of closure of union and intersection as above is problematic since the computer has finite capacity. Thus, if there is just sufficient memory to store a polytope  $O_i$ , then forming the union  $O_1 \cup O_2$  will not be possible in practice. In this discussion, it is assumed that the memory capacity is unlimited. This does not invalidate the previous paragraph, since the argument is that the number of possible definitions is finite, not the capacity for storing definitions.

Thus it has been shown that the set of polytopes defined above form the basis for a topological space. Note that this is not a metric space, since the requirement for density of points is not met. It does, however, closely parallel a metric space, and there exists a homomorphic mapping between.

## 2.2 Regular Polytopes as “Regular” Sets

Since the polytopes as defined above have been shown to form the basis of a topological space (using the “Open sets” axioms), they by definition are open sets, therefore the interior  $O^O$  of  $O$  is  $O$  itself – i.e.  $O^O = O$ .

It has been shown above that the inverse of a polytope is also a polytope. Since the inverse of an open set is closed, this shows that all polytopes are closed, therefore the closure of any polytope is the set itself -  $O^C = O$ . These together mean that the interior of the closure of a polytope is the polytope itself. Thus the polytopes are regular sets by definition.

Note that exactly the same logic as above could have been used to show that the regular polytopes satisfy the closed set axioms (C.1, C.2 and C.3) (Gaal 1964). Thus the argument is symmetric in terms of the concepts “open” and “closed”.

This may seem to be an unusual property – that the regular polytope is both open and closed, and this is indeed a much stronger assertion than that they are “open-closed” or “closed-open” sets. The “open\_closed” or “closed-open” sets are simply those sets which are neither open nor closed (partly closed, partly open, for example in 1D, the “half open” interval  $[0,1)$  – the real numbers  $x$  such that  $0 \leq x < 1$ ). The assertion on the regular polytope is that they are both fully open and fully closed.

This would clearly not be possible in a metric space, since there the inverse of a closed set cannot be closed (except for the empty set and the universal set). It is only possible because the set of all points in the computer representation is finite. Returning to the 1D example above, the dr\_rational numbers  $x$  such that  $0 \leq x < 1$  has a largest number (say  $x'$ ) such that  $x' < 1$ , but for all  $x < 1$ ,  $x \leq x'$ . In a similar way, the definition of half spaces (the four part definition – with the careful definition of the containment of points which fall on the boundary plane) ensures that there is a clean partition of the space of representational points.

### 3 Relationship between Half Space Equality and Pseudo-rational Point Set Equality.

#### 3.1 Point Set Equality

Two half spaces can be defined as equal if they define the same set of (domain-restricted rational) points. i.e.

$$H = H' \text{ is defined as } (p \in H) \Leftrightarrow (p \in H')$$

#### 3.2 Half Space Equality

Two half spaces  $H(A,B,C,D)$ ,  $H'(A',B',C',D')$  can be defined as equal if there exists a rational number  $r$  such that  $A=rA'$ ,  $B=rB'$ ,  $C=rC'$ ,  $D=rD'$ ,  $r>0$ . This is known as half-space equality and represented below as  $H \cong H'$ .

#### 3.3 Sufficiency of Half Space Equality

The half-space test for equality is sufficient to show pseudo-rational point set equality. That is to say  $H \cong H' \Rightarrow H = H'$ .

##### 3.3.1 Outline of Proof

The proof proceeds by showing that a point which satisfies the definition equation of  $H$ , must necessarily satisfy the equation for  $H'$ , and the reverse is also true.

##### 3.3.2 Proof

Assume  $H(A,B,C,D) \cong H'(A',B',C',D')$  – this means that there exists a rational number  $r$  such that  $A=rA'$ ,  $B=rB'$ ,  $C=rC'$ ,  $D=rD'$ ,  $r>0$ .

Let  $p$  be the relationship in the definition of  $H$  i.e.  $H(A,B,C,D)$  is defined as  $Ax+By+Cz \geq 0$ , and  $H'(A',B',C',D')$  is defined as  $A'x+B'y+C'z \geq 0$

Thus if  $A>0$ , or  $(A=0,B>0)$  or  $(A=0,B=0,C>0)$  or  $(A=0,B=0,C=0,D>0)$ ,  $p$  is  $\geq$ .

If  $A<0$ , or  $(A=0,B<0)$  or  $(A=0,B=0,C<0)$  or  $(A=0,B=0,C=0,D<0)$ ,  $p$  is  $>$ .

Assume dr\_rational point  $p = (x,y,z) \in H$

By definition this means that  $Ax+By+Cz \geq 0$ .

$$rA'x+rB'y+rC'z \geq 0$$

$$\text{or } A'x+B'y+C'z \geq 0 \text{ (dividing by } r \text{ – since } r > 0 \text{).}$$

Since  $A=rA'$ ,  $B=rB'$ ,  $C=rC'$ ,  $D=rD'$ ,  $r>0$ , it follows that  $p \in H'$ .

Therefore  $A'x+B'y+C'z \geq 0$ , or  $p \in H'$ .

Similar reasoning, multiplying by  $r$  instead of dividing shows that  $p \in H' \Rightarrow p \in H$ .

**Thus  $H \cong H' \Rightarrow H = H'$ .**

### 3.4 Necessity of Half Space Equality

The converse is not necessarily true unless the half space definition is constrained, since a half space could be defined as, for example  $H = H(1,1,1,-3M)$  (see Figure 4). This contains one point only,  $(M, M, M)$ , since point coordinate values are constrained to  $-M \leq x,y,z \leq M$ . The half space  $H(2,1,1,-4M)$  also contains this one point only  $(M, M, M)$ , and so  $H(1,1,1,-3M) = H(2,1,1,-4M)$  (point-set equality). These half spaces would not satisfy “half-space equality”.

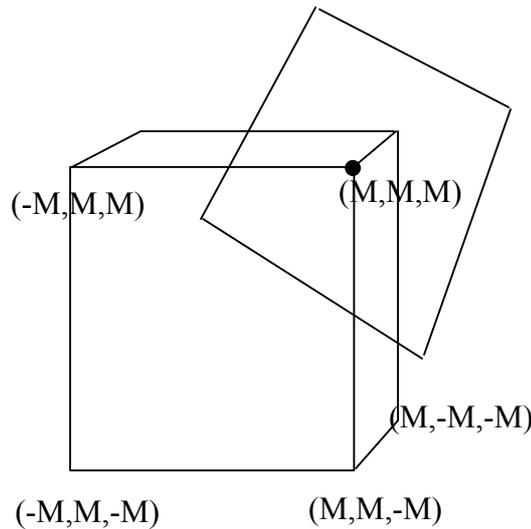


Figure 4 Half space defining one point only.

However, it is possible restrict the definition of half space to avoid these cases by requiring “well behaved half spaces”

**A well defined half space is defined as:**

**$H=H(A,B,C,D)$  is well defined if  $A=B=C=0$  ( $D \neq 0$ ) or there exists at least one pseudo-rational point  $(x,y,z)$ , such that  $Ax + By + Cz + D = 0$  with  $-M+1 \leq x,y,z \leq M-1$ ,**

That is to say, the universal and empty half planes are valid, otherwise the plane defining the half space must have at least one point within the cube  $-M+1 \leq x,y,z \leq M-1$ . note that

$H(0,0,0,D)$ ,  $D < 0$  is an empty half plane ( $\cong H_0$ ).

$H(0,0,0,D)$ ,  $D > 0$  is a universal half plane ( $\cong H_\infty$ ).

In the following proof:

for  $H = H(A,B,C,D)$ ,  $b(x,y,z)$  is defined as the value of  $Ax+By+Cz+D$ ,  $\rho$  is the relationship in the definition of  $H$ . That is  $H(A,B,C,D)$  is defined by  $Ax+By+Cz \rho 0$ , or equivalently  $b(x,y,z) \rho 0$ .

for  $H' = H(A',B',C',D')$ ,  $b'(x,y,z)$  is defined as  $A'x+B'y+C'z+D'$ ,  $\rho$  is the relationship in the definition of  $H'$  i.e.  $H(A',B',C',D')$  is defined by  $A'x+B'y+C'z \rho' 0$ , or equivalently  $b'(x,y,z) \rho' 0$ .

The notation  $sign(A) = +1$  if  $A>0$ ,  $-1$  if  $A<0$ ,  $0$  if  $A=0$ .

If  $H=H(A,B,C,D)$  and  $H'=H(A',B',C',D')$  are well defined half planes,  $H = H' \Rightarrow H \cong H'$ .

### 3.4.1 Outline of Proof

The first step of the proof is to show that  $H = H'$  implies that

$$\text{sign}(A) = \text{sign}(A').$$

$$\text{sign}(B) = \text{sign}(B').$$

$$\text{sign}(C) = \text{sign}(C').$$

$$\text{sign}(D) = \text{sign}(D').$$

$$\rho = \rho'.$$

The next step is to show that there exist 3 non collinear points that are on the plane defining  $H$ , and also on the plane defining  $H'$ . (Note, that this need not be true of planes which are not well defined, such as the one shown in Figure 4).

From this it follows that the planes defining  $H$  and  $H'$  are coplanar, and therefore their defining relationships are redundant.

### 3.4.2 Proof

Consider  $H(A,B,C,D)$ , and  $H'(A',B',C',D')$ , well defined halfspaces restricted as above such that  $\exists p=(x,y,z)$  st  $Ax+By+Cz+D=0$ ,  $-M+1<x,y,z<M-1$ ; and  $\exists p'=(x',y',z')$  st  $A'x'+B'y'+C'z'+D'=0$ ,  $-M+1<x',y',z'<M-1$ .

Assume that  $\forall p \in H, p \in H'$ , and  $\forall p' \in H', p' \in H$ . (i.e. assume  $H = H'$ )

### 3.4.3 Lemma 1: If $p \in H(A,B,C,D) \Leftrightarrow p \in H'(A',B',C',D')$ , $\text{sign}(A) = \text{sign}(A')$

**Assume  $A > 0$ .**

Consider any point  $p=(x,y,z)$  such that  $h(x,y,z) = Ax+By+Cz+D=0$ ,  $-M+1<x,y,z<M-1$ .

Consider  $p^+ = (x^+,y,z)$ . where  $x^+$  is a pseudo-rational point  $x < x^+ < M$ .<sup>3</sup>

$$h(x^+,y,z) = Ax^++By+Cz+D > 0, \text{ therefore } p^+ \in H.$$

$$\text{Thus } p^+ \in H' \text{ and so } h'(x^+,y,z) = A'x^++B'y+C'z+D' \geq 0$$

Consider  $p^- = (x^-,y,z)$ , where  $x^-$  is a pseudo-rational point  $-M < x^- < x$ .

$$A(x^-)+By+Cz+D < 0, \text{ therefore } p^- \notin H.$$

$$\text{Thus } p^- \notin H' \text{ and so } h'(x^-,y,z) = A'x^-+B'y+C'z+D' \leq 0$$

$$\text{Therefore } h'(x^-,y,z) - h'(x^+,y,z) \geq 0$$

$$\Rightarrow A'x^+ - A'(x^-) \geq 0.$$

$$\Rightarrow A'(x^+ - x^-) \geq 0.$$

$$\Rightarrow A' \geq 0. \text{ (since } x^+ > x^-).$$

Assume  $A' = 0$ ,

$$h'(x^+,y,z) = A'x^++B'y+C'z+D' = A'x^-+B'y+C'z+D' = h'(x^-,y,z)$$

$$\text{But } p^- \notin H'$$

$$\Rightarrow h'(x^-,y,z) < 0,$$

$$\Rightarrow h'(x^+,y,z) < 0,$$

<sup>3</sup> It is always possible to find such a number since  $x \leq M-1$ . For example,  $(2M+1)/2$  is such a number.

$p^+ \notin H'$  – contradiction!

Therefore  $A > 0 \Rightarrow A' > 0$ .

By the same reasoning  $A < 0 \Rightarrow A' < 0$ .

Applying the above in reverse,  $A' > 0 \Rightarrow A > 0$  and  $A' < 0 \Rightarrow A < 0$ .

Therefore it follows that  $A = 0 \Leftrightarrow A' = 0$ .

### 3.4.4 Lemma 2: If $p \in H(A, B, C, D) \Leftrightarrow p \in H'(A', B', C', D')$ , $sign(B) = sign(B')$

**Assume  $B > 0$ .**

Consider any point  $p(x, y, z)$  such that  $h(x, y, z) = 0$ ,  $-M < x, y, z < M$ .

Consider  $p^+ = (x, y^+, z)$ , where  $y^+$  is a pseudo-rational point  $y < y^+ < M$ .

$h(x, y^+, z) = A^*x + B^*y^+ + C^*z + D > 0$ , therefore  $p^+ \in H$ .

Thus  $p^+ \in H'$  and so  $h'(x, y^+, z) = A'x + B'y^+ + C'z + D \geq 0$

Consider  $p^- = (x, y^-, z)$ , where  $y^-$  is a pseudo-rational point  $-M < y^- < y$ .

$Ax + By^- + Cz + D < 0$ , therefore  $p^- \notin H$ .

Thus  $p^- \notin H'$  and so  $h'(x, y^-, z) \leq 0$

As above,  $B' \geq 0$ .

Assume  $B' = 0$ ,

$h'(x, y^+, z) = A'x + B'y^+ + C'z + D = A'x + B'y^- + C'z + D = h'(x, y^-, z)$

But  $p^- \notin H'$

$\Rightarrow h'(x, y^-, z) < 0$ ,

$\Rightarrow h'(x, y^+, z) < 0$ ,

$p^+ \notin H'$  – contradiction!

Therefore  $B > 0 \Rightarrow B' > 0$ .

As before this can be applied in reverse.

Giving  $sign(B) = sign(B')$ .

### 3.4.5 Lemma 3: If $p \in H(A, B, C, D) \Leftrightarrow p \in H'(A', B', C', D')$ , $sign(C) = sign(C')$

The same reasoning applies for C.

Thus  $sign(A) = sign(A')$ ,  $sign(B) = sign(B')$ ,  $sign(C) = sign(C')$

### 3.4.6 Lemma 4: If $p \in H(A, B, C, D) \Leftrightarrow p \in H'(A', B', C', D')$ , then $\rho = \rho'$

If  $A > 0$ , then  $A' > 0$  therefore  $\rho$  is  $\geq$ , and  $\rho'$  is  $\geq$ , i.e  $\rho = \rho'$ .

If  $A < 0$ , then  $A' < 0$  therefore  $\rho$  is  $>$ , and  $\rho'$  is  $>$ , i.e  $\rho = \rho'$ .

If  $A = 0$ ,  $B > 0$ , then  $A' = 0$ ,  $B' > 0$  therefore  $\rho$  is  $\geq$ , and  $\rho'$  is  $\geq$ , i.e  $\rho = \rho'$ .

If  $A = 0$ ,  $B < 0$ , then  $A' = 0$ ,  $B' < 0$  therefore  $\rho$  is  $>$ , and  $\rho'$  is  $>$ , i.e  $\rho = \rho'$ .

And so on.

### 3.5 Proof that $(p \in H \Leftrightarrow p \in H') \Rightarrow H \cong H'$

Without loss of generality, assume  $|B| \geq |A|$  and  $|B| \geq |C|$ . (If this is not the case, the following argument can be applied replacing B and y with A and x or C and z whichever is the larger coefficient. The reason for choosing B and y for the main argument rather than A and x is that some of the cases below cannot apply to A and x. These cases are made impossible because the sign of A determines the type of relational operator).

As noted above, the statement that  $p \in H(A,B,C,D)$ ,  $p = (x,y,z)$  can be written as  $h(x,y,z) \rho 0$ ,

where  $\rho$  is  $\geq$  if  $A > 0$  or  $A = 0, B > 0$  or  $A = B = 0, C > 0$  or  $A = B = C = 0, D > 0$

$\rho$  is  $>$  if  $A < 0$  or  $A = 0, B < 0$  or  $A = B = 0, C < 0$  or  $A = B = C = 0, D < 0$

Five cases need to be considered:

1.  $B > 0$ ,  $\rho$  is  $\geq$
2.  $B < 0$ ,  $\rho$  is  $\geq$
3.  $B > 0$ ,  $\rho$  is  $>$
4.  $B < 0$ ,  $\rho$  is  $>$
5.  $B = 0$ .

#### 3.5.1 Case 1 $B > 0$ , $\rho$ is $\geq$

Lemma 2 (or 1, or 3) shows that  $B' > 0$ , by lemma 4,  $\rho'$  is  $\geq$ .

Consider point  $p = (x,y,z)$ : on the defining plane of  $H$ .

$$\text{i.e. } h(x,y,z) = Ax + By + Cz + D = 0, -M + 1 \leq x, y, z \leq M - 1. (1)$$

Adjust the position of this point to  $p_1 = (x_1, y_1, z_1)$  where:

$$x_1 = x + \delta_x \text{ such that } x_1 = \frac{X}{4B'M}, 0 < \delta_x \leq \frac{1}{4B'M}, X \in I. (2)$$

$$z_1 = z + \delta_z \text{ such that } z_1 = \frac{Z}{4B'M}, 0 < \delta_z \leq \frac{1}{4B'M}, Z \in I. (3)$$

(That is to say, using a grid of size  $1/4B'M$ , find the nearest larger  $x$  and  $z$  values that fall on the grid).

$$y_1 = y - \frac{A\delta_x + C\delta_z}{B} (4)$$

It is clear that  $-M < x_1, z_1 < M$ , and that  $x_1$  and  $z_1$  are pseudo-rational.

$$\left| \frac{A\delta_x + C\delta_z}{B} \right| \leq \left| \frac{A}{4B'M} \right| + \left| \frac{C}{4B'M} \right| \leq \frac{1}{2M} \text{ (since } B, B' > 0 \text{ and } |A|, |C| \leq |B|).$$

$$\text{Therefore } -M + 1/2 \leq y_1 \leq M - 1/2 (5)$$

Note that  $h(x_1, y_1, z_1) = Ax_1 + By_1 + Cz_1 + D$

$$= A(x + \delta_x) + By - B \frac{A\delta_x + C\delta_z}{B} + C(z + \delta_z) + D \text{ (substituting 2, 3 & 4)}$$

$$= Ax + By + Cz + D + A\delta_x - B \frac{A\delta_x + C\delta_z}{B} + C\delta_z$$

$$= A\delta_x - (A\delta_x + C\delta_z) + C\delta_z \quad (\text{since } Ax+By+Cz+D=0)$$

$$= 0.$$

$$\text{Thus } h(x_1, y_1, z_1) = Ax_1 + By_1 + Cz_1 + D = 0 \quad (6)$$

$$\text{Therefore } y_1 = -\frac{Ax_1 + Cz_1 + D}{B} \quad (\text{re-arranging 6})$$

$$= -\frac{AX + CZ}{B'M} - \frac{D}{B}$$

$$= -\frac{AX + CZ + B'DM}{B'M}$$

Now  $B'M$  is an integer  $|B'M| < 6M^3$ .

and  $|AX+CZ+B'DM| < 3M^3 < 6M^4$ .

Therefore  $y_1$  is a pseudo-rational number.

Therefore  $p_1$  is a pseudo-rational point on the defining plane of  $H$ .

Since  $\rho$  is  $\geq 0$ ,  $h(x_1, y_1, z_1) = 0 \Rightarrow p_1 \in H$  and therefore  $p_1 \in H'$ .

Therefore  $h'(x_1, y_1, z_1) \geq 0$ .

Let  $h'(x_1, y_1, z_1) = k$ . ( $k \geq 0$ ).

$$\text{i.e. } k = A'x_1 + B'y_1 + C'z_1 + D' \quad (6a)$$

$$= A' \frac{X}{4B'M} + B'y_1 + C' \frac{Z}{4B'M} + D'$$

$$= B'y_1 + \frac{A'X + C'Z + 4B'MD'}{4B'M} \quad (7)$$

$$\text{Let } y' \text{ be defined as a (rational) number } y' = -\frac{A'x_1 + C'z_1 + D'}{B'} \quad (7a)$$

$$\text{It follows that } A'x_1 + B'y' + C'z_1 + D' = 0 \quad (\text{by simple substitution}) \quad (7b)$$

$$\text{but } A'x_1 + B'y_1 + C'z_1 + D' = k \quad (\text{from 6a})$$

$$\text{therefore } B'(y_1 - y') = k \quad (8)$$

$$\text{Now } y' = -\frac{A'X + C'Z + 4B'MD'}{4B'^2M} \quad (\text{substituting 2 and 3 into 7a})$$

but  $4B'^2M$  is an integer  $|4B'^2M| < 6M^3$ .

and  $|A'A + C'Z + 4B'MD'|$  is an integer  $< 3M^3 < 6M^4$ .

$$\text{Clearly } y' \leq y_1 \quad (\text{since rearranging (8) gives } y' = y_1 - k/B') \quad (8a)$$

therefore  $y' < M$ .

Let  $\tilde{p} = (x_1, -M, z_1)$ .

$h(x_1, -M, z_1) < h(x_1, y_1, z_1)$  (since  $-M < y_1$ ,  $B > 0$ )

therefore  $h(x_1, -M, z_1) < 0$  (by 1)

therefore  $\tilde{p} \notin H$  and  $\tilde{p} \notin H'$ .

therefore  $h'(x_1, -M, z_1) < 0$

and hence  $-M < y'$  (from  $h'(x_1, -M, z_1) < 0$ ,  $h'(x_1, y', z_1) = 0$ , and  $B' > 0$ )

therefore  $y'$  is a pseudo-rational number,  $|y'| < M$

Thus, using  $A'x_1 + B'y' + C'z_1 + D' = 0$  (from 7a, and  $B' > 0$ ).

Therefore  $(x_1, y', z_1) \in H'$  and so  $(x_1, y', z_1) \in H$ .

It follows that  $Ax_1 + By' + Cz_1 + D \geq 0$

but  $Ax_1 + By_1 + Cz_1 + D = 0$  (from 6)

therefore  $B(y' - y_1) \geq 0$  (9)

and since  $B > 0$ ,  $y' \geq y_1$ .

with (8a) this shows that  $y' = y_1$ .

Thus point  $p_1$  is on the defining plane of  $H$  and of  $H'$ .

The above argument holds for the point  $p_2 = (x_2, y_2, z_2)$  where:

$$x_2 = \frac{X-1}{4B'M}$$

let  $\delta_{x_2} = x_2 - x$ .

$$= \frac{X-1}{4B'M} - x$$

$$= x_1 - x - \frac{1}{4B'M}$$

$$= \delta_x - \frac{1}{4B'M}$$

$$\text{but } 0 < \delta_x \leq \frac{1}{4B'M} \Rightarrow -\frac{1}{4B'M} < \delta_{x_2} \leq 0$$

So  $x_2 = x + \delta_{x_2}$  where  $-\frac{1}{4B'M} < \delta_{x_2} \leq 0$

$$z_2 = z_1$$

$$y_2 = y - \frac{A\delta_{x_2} + C\delta_z}{B} \quad (12)$$

The same reasoning shows that point  $p_2$  is on the defining plane of  $H$  and therefore  $H'$ .

The above argument also holds for the point  $p_3 = (x_3, y_3, z_3)$  where:

$$x_3 = x_1$$

$$z_3 = \frac{Z-1}{4B'M}$$

let  $\delta_{z_3} = z_3 - z$ .

$$z_3 = z + \delta_{z_3}, -\frac{1}{4B'M} < \delta_{z_3} \leq 0 \text{ as above.}$$

$$y_3 = y - \frac{A\delta_x + C\delta_{z_3}}{B} \quad (13)$$

The same reasoning shows that point  $p_3$  is on the defining plane of  $H$  and of  $H'$ .

From above we have three points on both the planes defining H and H'. these points are:

$$p_1=(x_1,y_1,z_1)$$

$$p_2=(x_2,y_2,z_2) = (x_1-\delta,y_2,z_1)$$

$$p_3=(x_3,y_3,z_3) = (x_1,y_3,z_1-\delta)$$

$$\text{where } \delta = \frac{1}{4B'M}$$

Substituting – we get:

$$Ax_1+By_1+Cz_1+D=0 \quad (15)$$

$$A'x_1+B'y_1+C'z_1+D=0 \quad (16)$$

$$A(x_1-\delta)+By_2+Cz_1+D=0 \quad (17)$$

$$A'(x_1-\delta)+B'y_2+C'z_1+D=0 \quad (18)$$

$$Ax_1+By_3+C(z_1-\delta)+D=0 \quad (19)$$

$$A'x_1+B'y_3+C'(z_1-\delta)+D=0 \quad (20)$$

Combining, we get:

$$B(y_1-y_2)+A\delta = 0 \quad (\text{subtracting 17 from 15})$$

$$\text{or } \frac{A}{B} = \frac{y_2 - y_1}{\delta} \quad (\text{since } B>0, \delta>0)$$

and from 18 and 15, it follows that  $\frac{A'}{B'} = \frac{y_2 - y_1}{\delta}$

$$\text{Thus } \frac{A}{B} = \frac{A'}{B'} \quad \text{let } a = \frac{A}{B} = \frac{A'}{B'}$$

Using equations 15, 16, 19 and 20, it follows that

$$\frac{C}{B} = \frac{y_3 - y_1}{\delta} = \frac{C'}{B'} \quad \text{let } c = \frac{C}{B} = \frac{C'}{B'}$$

Dividing 15 by B, and 16 by B' we get:

$$ax_1+y_1+cz_1+D/B=0$$

$$a'x_1+y_1+c'z_1+D'/B'=0$$

$$\text{leading to: } \frac{D}{B} = \frac{D'}{B'}$$

Let  $r=B/B'$ , it is readily seen that

$$A=rA'$$

$$B=rB'$$

$$C=rC'$$

$$\text{and } D=rD'$$

**Since  $B>0$  and  $B'>0$ ,  $r>0$  therefore  $H(A,B,C,D) \cong H'(A',B',C',D')$ .**

### 3.5.2 Case 2 $B < 0$ , $\rho$ is $\geq$

Lemma 2 (or 1, or 3) show that  $B' < 0$ , by lemma 4, the relation defining  $H'$  is  $\geq$ .  
**Consider point  $p=(x,y,z)$ : on the defining plane of  $H$ .**

$$\text{i.e. } h(x,y,z) = Ax+By+Cz+D=0, -M+1 \leq x,y,z \leq M-1. \quad (21)$$

As in case 1 adjust the position of this point to  $p_1 = (x_1, y_1, z_1)$  where:

$$x_1 = x + \delta_x \text{ such that } x_1 = \frac{X}{-4B'M}, 0 < \delta_x \leq \frac{1}{-4B'M}, X \in I. \quad (22)$$

$$z_1 = z + \delta_z \text{ such that } z_1 = \frac{Z}{-4B'M}, 0 < \delta_z \leq \frac{1}{-4B'M}, Z \in I.. \quad (23)$$

$$y_1 = y + \frac{A\delta_x + C\delta_z}{-B} \quad (24)$$

It is clear that  $-M < x_1, z_1 < M$ , and that  $x_1$  and  $z_1$  are pseudo-rational.

$$\left| \frac{A\delta_x + C\delta_z}{-B} \right| \leq \left| \frac{A}{4B'M} \right| + \left| \frac{C}{4B'M} \right| \leq \frac{1}{2M} \text{ (since } |B|, |B'| > 0 \text{ and } |A|, |C| \leq |B|).$$

$$\text{Therefore } -M + \frac{1}{2} \leq y_1 \leq M - \frac{1}{2} \quad (25)$$

Note that  $h(x_1, y_1, z_1) = Ax_1 + By_1 + Cz_1 + D$

$$\begin{aligned} &= A(x + \delta_x) + By + B \frac{A\delta_x + C\delta_z}{-B} + C(z + \delta_z) + D \quad (\text{substituting 22,23 \&} \\ &\quad 24) \\ &= Ax + By + Cz + D + A\delta_x - B \frac{A\delta_x + C\delta_z}{B} + C\delta_z \\ &= A\delta_x - (A\delta_x + C\delta_z) + C\delta_z \quad (\text{since } Ax+By+Cz+D=0) \\ &= 0. \end{aligned}$$

$$\text{Thus } h(x_1, y_1, z_1) = Ax_1 + By_1 + Cz_1 + D = 0 \quad (26)$$

$$\text{Therefore } y_1 = \frac{Ax_1 + Cz_1 + D}{-B} \text{ (re-arranging 26)}$$

$$\begin{aligned} &= -\frac{AX + CZ}{B'M} - \frac{D}{B} \\ &= -\frac{AX + CZ + B'DM}{B'M} \end{aligned}$$

Now  $B'M$  is an integer  $0 < B'M < 6M^3$ . (since  $\text{sign}(B) = \text{sign}(B')$ )

and  $|AX + CZ + B'DM| < 3M^3 < 6M^4$ .

Therefore  $y_1$  is a pseudo-rational number.

Therefore  $p_1$  is a pseudo-rational point on the defining plane of  $H$ .

Since  $\rho$  is  $\geq 0$ ,  $h(x_1, y_1, z_1) = 0 \Rightarrow p_1 \in H$  and therefore  $p_1 \in H'$ .

Therefore  $h'(x_1, y_1, z_1) \geq 0$ .

Let  $h'(x_1, y_1, z_1) = k$ . ( $k \geq 0$ ).

$$\text{i.e. } k = A'x_1 + B'y_1 + C'z_1 + D' \quad (26a)$$

$$= A' \frac{X}{4B'M} + B'y_1 + C' \frac{Z}{4B'M} + D'$$

$$= B'y_1 + \frac{A'X + C'Z + 4B'MD'}{4B'M} \quad (27)$$

Let  $y'$  be defined as a (rational) number  $y' = \frac{A'x_1 + C'z_1 + D'}{-B'}$

$$\text{It follows that } A'x_1 + B'y' + C'z_1 + D' = 0 \text{ (by simple substitution)} \quad (27a)$$

$$\text{but } A'x_1 + B'y_1 + C'z_1 + D' = k \text{ (from 6a)}$$

$$\text{therefore } B'(y_1 - y') = k \quad (28)$$

$$\text{Now } y' = -\frac{A'X + C'Z + 4B'MD'}{4B'^2 M} \text{ (substituting 22 and 23 into 37a)}$$

$$\text{but } 4B'^2 M \text{ is an integer } 0 < 4B'^2 M < 6M^3.$$

$$\text{and } |A'A + C'Z + 4B'MD'| \text{ is an integer } < 3M^3 < 6M^4.$$

$$\text{Clearly } y' \geq y_1 \text{ (since rearranging (28) gives } y' = y_1 + k/(-B') \text{)} \quad (28a)$$

$$\text{therefore } y' > -M.$$

Let  $p^+ = (x_1, M, z_1)$ .

$$h(x_1, M, z_1) < h(x_1, y_1, z_1) \text{ (since } y_1 < M, B < 0)$$

$$\text{therefore } h(x_1, M, z_1) < 0 \text{ (by 1)}$$

$$\text{therefore } p \notin H \text{ and } p \notin H'$$

$$\text{therefore } h'(x_1, M, z_1) < 0$$

$$\text{and hence } y' < M \text{ (from } h'(x_1, M, z_1) < 0, h'(x_1, y', z_1) = 0, \text{ and } B' < 0)$$

$$\text{therefore } y' \text{ is a pseudo-rational number, } |y'| < M$$

Thus, using  $A'x_1 + B'y' + C'z_1 + D' = 0$  (from 27a).

$$\text{Therefore } (x_1, y', z_1) \in H' \text{ and so } (x_1, y', z_1) \in H.$$

$$\text{It follows that } Ax_1 + By' + Cz_1 + D \geq 0$$

$$\text{but } Ax_1 + By_1 + Cz_1 + D = 0 \text{ (from 26)}$$

$$\text{therefore } B(y' - y_1) \geq 0 \quad (29)$$

$$\text{and since } B < 0, y' \leq y_1.$$

$$\text{with (28a) this shows that } y' = y_1.$$

Thus point  $p_1$  is on the defining plane of  $H$  and of  $H'$ .

The above argument holds for the point  $p_2 = (x_2, y_2, z_2)$  where:

$$x_2 = \frac{X-1}{4B'M}$$

$$\text{Or as above, } x_2 = x + \delta_{x_2} \text{ where } -\frac{1}{4B'M} < \delta_{x_2} \leq 0$$

$$z_2 = z_1$$

$$y_2 = y - \frac{A\delta_{x_2} + C\delta_z}{B} \quad (12)$$

The same reasoning shows that point  $p_2$  is on the defining plane of  $H$  and therefore  $H'$ .

The above argument holds for the point  $p_3 = (x_3, y_3, z_3)$  where:

$$x_3 = x_1$$

$$z_3 = \frac{Z-1}{4B'M}$$

Or as above,  $z_3 = z + \delta_{z_3}, -\frac{1}{4B'M} < \delta_{z_3} \leq 0$  as above.

$$y_3 = y - \frac{A\delta_x + C\delta_{z_3}}{B} \quad (13)$$

The same reasoning shows that point  $p_3$  is on the defining plane of  $H$  and of  $H'$ .

From above we have three points on both the planes defining  $H$  and  $H'$ . these points are:

$$p_1 = (x_1, y_1, z_1)$$

$$p_2 = (x_2, y_2, z_2) = (x_1 - \delta, y_2, z_1)$$

$$p_3 = (x_3, y_3, z_3) = (x_1, y_3, z_1 - \delta)$$

$$\text{where } \delta = \frac{1}{4B'M}$$

The remainder of the argument follows as per case 1.

**Therefore  $H(A,B,C,D) \cong H'(A',B',C',D')$ .**

### 3.5.3 Case 3 $B > 0$ , $\rho$ is $>$

Lemma 2 (or 1, or 3) show that  $B' > 0$ , by lemma 4, the relation defining  $H'$  is  $>$

**Consider point  $p = (x, y, z)$ : on the defining plane of  $H$ . as per case 1**

$$\text{i.e. } h(x, y, z) = Ax + By + Cz + D = 0, -M + 1 \leq x, y, z \leq M - 1. \quad (31)$$

Adjust the position of this point to  $p_1 = (x_1, y_1, z_1)$  where:

$$x_1 = x + \delta_x \text{ such that } x_1 = \frac{X}{4B'M}, 0 < \delta_x \leq \frac{1}{4B'M}, X \in I. \quad (32)$$

$$z_1 = z + \delta_z \text{ such that } z_1 = \frac{Z}{4B'M}, 0 < \delta_z \leq \frac{1}{4B'M}, Z \in I.. \quad (33)$$

(That is to say, using a grid of size  $1/4B'M$ , find the nearest larger  $x$  and  $z$  values that fall on the grid).

$$y_1 = y - \frac{A\delta_x + C\delta_z}{B} \quad (34)$$

As before,  $h(x_1, y_1, z_1) = Ax_1 + By_1 + Cz_1 + D = 0$  (36)

Therefore  $y_1 = -\frac{Ax_1 + Cz_1 + D}{B}$  (re-arranging 36)

and  $y_1$  is a pseudo-rational number.

Therefore  $p_i$  is a pseudo-rational point on the defining plane of  $H$ .  
 Since  $\rho$  is  $> 0$ ,  $h(x_1, y_1, z_1) = 0 \Rightarrow p_i \notin H$  and therefore  $p_i \notin H'$ .

Therefore  $h'(x_1, y_1, z_1) \leq 0$ .

Let  $h'(x_1, y_1, z_1) = k$ . ( $k \leq 0$ ).

$$\text{i.e. } k = A'x_1 + B'y_1 + C'z_1 + D' \quad (36a)$$

$$= A' \frac{X}{4B'M} + B'y_1 + C' \frac{Z}{4B'M} + D'$$

$$= B'y_1 + \frac{A'X + C'Z + 4B'MD'}{4B'M} \quad (37)$$

Let  $y'$  be defined as a (rational) number  $y' = -\frac{A'x_1 + C'z_1 + D'}{B'}$

$$\text{It follows that } A'x_1 + B'y' + C'z_1 + D' = 0 \text{ (by simple substitution)} \quad (37a)$$

$$\text{but } A'x_1 + B'y_1 + C'z_1 + D' = k \text{ (from 36a)}$$

$$\text{therefore } B'(y_1 - y') = k \quad (38)$$

$$\text{Now } y' = -\frac{A'X + C'Z + 4B'MD'}{4B'^2 M} \text{ (substituting 32 and 33 into 37a)}$$

but  $4B'^2 M$  is an integer  $|4B'^2 M| < 6M^3$ .

and  $|A'A + C'Z + 4B'MD'|$  is an integer  $< 3M^3 < 6M^4$ .

$$\text{Clearly } y' \geq y_1 \text{ (since rearranging (8) gives } y' = y_1 - k/B', \text{ and } k \leq 0, B > 0) \quad (38a)$$

therefore  $y' < M$ .

Let  $\tilde{p} = (x_1, -M, z_1)$ .

$h(x_1, -M, z_1) < h(x_1, y_1, z_1)$  (since  $-M < y_1, B > 0$ )

therefore  $h(x_1, -M, z_1) < 0$  (by 1)

therefore  $\tilde{p} \notin H$  and  $\tilde{p} \notin H'$ .

therefore  $h'(x_1, -M, z_1) \leq 0$

and hence  $-M \leq y'$  (from  $h'(x_1, -M, z_1) \leq 0, h'(x_1, y', z_1) = 0$ , and  $B' > 0$ )

therefore  $y'$  is a pseudo-rational number,  $|y'| \leq M$

Thus,  $A'x_1 + B'y' + C'z_1 + D' = 0$  (from 37a, and  $B' > 0$ ).

Therefore  $(x_1, y', z_1) \notin H'$  and so  $(x_1, y', z_1) \notin H$ .

It follows that  $Ax_1 + By' + Cz_1 + D \leq 0$

$$\text{but } Ax_1 + By_1 + Cz_1 + D = 0 \text{ (from 36)}$$

$$\text{therefore } B(y' - y_1) \leq 0 \quad (39)$$

and since  $B > 0, y' \leq y_1$ .

with (38a) this shows that  $y' = y_1$ .

Thus point  $p_i$  is on the defining plane of  $H$  and of  $H'$ .

The above argument holds for the point  $p_2 = (x_2, y_2, z_2)$  where:

$$x_2 = \frac{X-1}{4B'M}$$

Or as above,  $x_2 = x + \delta_{x_2}$  where  $-\frac{1}{4B'M} < \delta_{x_2} \leq 0$

$$z_2 = z_1$$

$$y_2 = y - \frac{A\delta_{x_2} + C\delta_z}{B} \quad (42)$$

The same reasoning shows that point  $p_2$  is on the defining plane of  $H$  and therefore  $H'$ .

The above argument holds for the point  $p_3 = (x_3, y_3, z_3)$  where:

$$x_3 = x_1$$

$$z_3 = \frac{Z-1}{4B'M}$$

$$\text{let } \delta_{z_3} = z_3 - z.$$

$$z_3 = z + \delta_{z_3}, \quad -\frac{1}{4B'M} < \delta_{z_3} \leq 0 \text{ as above.}$$

$$y_3 = y - \frac{A\delta_x + C\delta_{z_3}}{B} \quad (43)$$

The same reasoning shows that point  $p_3$  is on the defining plane of  $H$  and of  $H'$ .

From above we have three points on both the planes defining  $H$  and  $H'$ . these points are:

$$p_1 = (x_1, y_1, z_1)$$

$$p_2 = (x_2, y_2, z_2) = (x_1 - \delta, y_2, z_1)$$

$$p_3 = (x_3, y_3, z_3) = (x_1, y_3, z_1 - \delta)$$

$$\text{where } \delta = \frac{1}{4B'M}$$

The remainder of the argument follows as per case 1.

**Therefore  $H(A,B,C,D) \cong H'(A',B',C',D')$ .**

### 3.5.4 Case 3 $B < 0$ , $\rho$ is $>$

This follows in the same way as the above.

### 3.5.5 Case 4 $B = 0$

Since, by the assumption,  $|B| \geq |A|$  and  $|B| \geq |C|$ , it follows that  $A=0$ , and  $C=0$ , and therefore  $A'=0$ ,  $B'=0$ ,  $C'=0$ . also  $\rho' = \rho$ .

If  $D > 0$ , then for any valid  $p$ ,  $b(x,y,z) = D$ . Since  $D > 0$   $p \in H$

Thus  $p \in H'$ . this means  $A'x + B'y + C'z + D' > 0$ . i.e.  $D' > 0$ .

Since  $A=B=C=D=0$  is not allowed, it follows that  $D > 0 \Rightarrow D' > 0$ .

Applying this argument in reverse  $D' > 0 \Rightarrow D > 0$ .

Since  $D \neq 0$  and  $D' \neq 0$   $D' < 0 \Leftrightarrow D < 0$ .

Let  $r=D/D'$ , it is readily seen that

$$A=rA'$$

$$B=rB'$$

$$C=rC'$$

$$\text{and } D=rD'$$

Since  $D>0$  and  $D'>0$ ,  $r>0$  therefore  $H(A,B,C,D) \cong H(A',B',C',D')$ .

### 3.5.6 Conclusion

The assumption without loss of generality was that  $|B| \geq |A|$  and  $|B| \geq |C|$ . If this is not the case, the above arguments can be carried out by substituting A for B or C for B in the above proof (and visa versa). But note that if A is the largest coefficient, cases 2 and 3 cannot occur. (since  $A<0$  defines the relation  $\rho$  as  $>$ ).

Thus we can state that:

$$H(A,B,C,D) \cong H(A',B',C',D') \Rightarrow H(A,B,C,D) = H(A',B',C',D')$$

And for well defined half spaces,

$$H(A,B,C,D) \cong H(A',B',C',D') \Leftrightarrow H(A,B,C,D) = H(A',B',C',D').$$

## 4 Vertex Test for Redundancy

This following test allows a considerable simplification in the algorithms that can be written to manipulate regular polytopes, by reducing the special cases that need to be accommodated. The following lemma shows that a simplified containment test can be applied, rather than that specified in the half-space definition. Put simply – the details of the relation operator can be ignored, and a simple test  $\geq$  can be used. In Figure 5, the half plane  $H$  is redundant to the definition of  $C$ , even though it passes through a vertex.

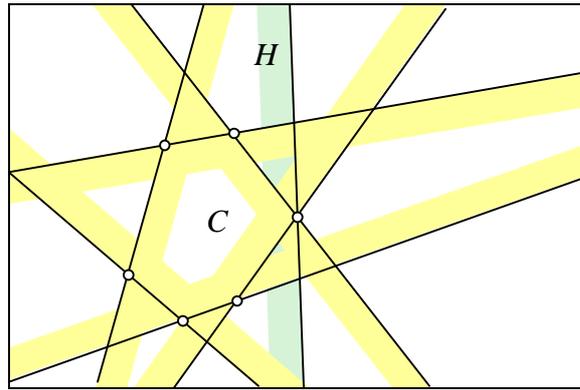


Figure 5 Vertex test for redundancy.

### 4.1 Lemma

If all points within the pseudo-closure of a convex polytope are within the pseudo-closure of a half plane, then the half plane is redundant to the definition of the convex polytope. That is there is no point within the convex polytope which does not fall within the half space.

#### 4.1.1 Outline of Proof

The half planes which define the convex polytope can be divided into “Eastern” and “Western” and others – i.e. those with  $A > 0$ , which mark the right side of the convex polytope, those with  $A < 0$ , and those with  $A = 0$ .

Eastern boundaries are those for which the defining relation relation is  $\geq$ ,

Assume  $H (\neq H^\infty)$  is such that all points  $p = (x,y,z) \in C^{\text{pc}}$  satisfy  $Ax+By+Cz+D \geq 0$ . i.e. all vertices are within the pseudo-closure of  $H$ .

If  $H$  is not redundant to  $C$ , then there must be some pseudo-rational point  $p \in C$ , such that  $p \notin H$ . (Otherwise the point-set definition of  $C$  is not affected by the addition of  $H$ ).

Thus

$$Ax+By+Cz+D < 0 \text{ if } A > 0, \text{ if } A=0, B > 0, \text{ or if } A=B=0, C > 0 \text{ or if } H=H^0, \text{ or} \quad (1)$$

$$Ax+By+Cz+D >= 0 \text{ if } A < 0, \text{ if } A=0, B < 0, \text{ or if } A=B=0, C < 0 \text{ or if } H=H^\infty \quad (2)$$

The first of these is impossible, since  $Ax+By+Cz+D \geq 0$ . Therefore it must follow that

$$Ax+By+Cz+D = 0 \quad (3)$$

$$\text{and } A < 0, \text{ or } A=0, B < 0, \text{ or } A=B=0, C < 0 \text{ or } H=H^\infty. \quad (4)$$

Since  $p \in C$ . then for all  $H_j (A_j, B_j, C_j, D_j) \in C$ ,

$$A_j x + B_j y + C_j z + D \geq 0 \text{ if } A_j > 0, \text{ if } A_j = 0, B_j > 0, \text{ or if } A_j = B_j = 0, C_j > 0. \quad (5)$$

$$A_j x + B_j y + C_j z + D > 0 \text{ if } A_j < 0, \text{ if } A_j = 0, B_j < 0, \text{ or if } A_j = B_j = 0, C_j < 0. \quad (6)$$

Let  $\mathbf{H}^f$  be the set  $\{H_j (A_j, B_j, C_j, D_j), H_j \in C, A_j < 0\}$  (i.e the subset of the half planes defining C for which  $A_j < 0$ ).

If  $\mathbf{H}^f$  is empty, let  $p^+ = (x^+, y, z)$  where  $x^+$  is any pseudo-rational number  $x < x^+ < M$ .

$$\text{for all } H_j (A_j, B_j, C_j, D_j) \in C, A_j \geq 0. \quad (7)$$

$$\text{Therefore } A_j x^+ + B_j y + C_j z + D \geq A_j x + B_j y + C_j z + D \geq 0. \quad (8)$$

therefore  $p^+ \in C^{pc}$

If  $\mathbf{H}^f$  is non-empty,

For  $H_j (A_j, B_j, C_j, D_j) \in \mathbf{H}^f$ ,

$$\text{Let } x_j = \frac{B_j y + C_j z + D}{-A_j}. \text{ (} x_j \text{ is pseudo-rational)} \quad (9)$$

$$A_j x_j + B_j y + C_j z + D = 0. \quad (10)$$

Since  $A_j < 0$ , and  $(x_j, y, z) \in H_j$ ,

$$A_j x + B_j y + C_j z + D > 0 \text{ (from 6)} \quad (11)$$

$$\text{therefore } x_j > x. \text{ (from 11 and 10)} \quad (12)$$

Let  $x^+ = \min_{H_j \in \mathbf{H}^f} (x_j)$ . (That is, the minimum of all  $x_j$  for which  $A_j < 0$ )

Consider the point  $p^+ = (x^+, y, z)$ .

For  $H_j (A_j, B_j, C_j, D_j) \in \mathbf{H}^f$ , since  $x^+ \leq x_j$ , and  $A_j < 0$ ,

$$A_j x^+ + B_j y + C_j z + D \geq A_j x_j + B_j y + C_j z + D = 0, \quad (13)$$

$$\text{therefore } p^+ \in H_j^{pc} \text{ for } H_j \in \mathbf{H}^f, \quad (14)$$

For  $H_j (A_j, B_j, C_j, D_j) \notin \mathbf{H}^f$ ,

since  $A_j \geq 0$  and  $x^+ > x$ ,

$$A_j x^+ + B_j y + C_j z + D \geq A_j x + B_j y + C_j z + D \geq 0 \quad (15)$$

$$\text{therefore } p^+ \in H_j^{pc} \text{ for } H_j \notin \mathbf{H}^f \quad (16)$$

therefore  $p^+ \in C^{pc}$  (by 14 and 16)

In both these cases,  $p^+ \in C^{pc}$  leads to  $Ax^+ + By + Cz + D \geq 0$ .

but  $A \leq 0$  (from 4) and  $x^+ > x$  and  $Ax + By + Cz + D = 0$  (from 3) imply that  $A=0$ , and therefore  $B < 0$ , or  $B=0, C < 0$  or  $H=H^\infty$ .

If  $A=0, B < 0$

Let  $H^c$  be the set  $\{H_j (A_j, B_j, C_j, D_j), H_j \in C, A_j < 0\}$  (i.e the subset of the half planes defining C for which  $A_j < 0$ ).

Let  $H^w$  be the set  $\{H_j (A_j, B_j, C_j, D_j), H_j \in C, A_j > 0\}$  (i.e the subset of the half planes defining C for which  $A_j > 0$ ).

Let  $H_i$  be the half space  $H_i \in H^w$  which passes through  $p$  with the largest value of  $B_i/A_i$  (could be worded better)

Define  $p^+$  as above.

for each  $H_j \in H^c$  calculate  $p_{ij}$  as the point of intersection of  $H_i$  and  $H_j$  (not necessarily a pseudo-rational point

$$A_i x_{ij} + B_i y_{ij} + C_i z + D_i = 0$$

$$A_j x_{ij} + B_j y_{ij} + C_j z + D_j = 0$$

$$y_{ij} = -\frac{(C_i z + D_i)A_j - (C_j z + D_j)A_i}{B_i A_j - B_j A_i}$$

$$x_{ij} = \frac{(C_i z + D_i)B_j - (C_j z + D_j)B_i}{B_i A_j - B_j A_i}$$

for each of  $H_j$  which intersects with  $H_i$  in this plane – i.e those for which there exists  $(x_j, y_j, z)$  such that  $A_j X_j + B_j Y_j + C_j z + D_j = 0$ , and  $A_i X_i + B_i Y_i + C_i z + D_i = 0$  let  $p_j = (x_j, y_j, z)$ . where this doesn't exist let  $p_j = (x_j, M, z)$  (where  $H_i$  meets  $x=M$ ).

choose  $p'$  as the  $p_j$  with the minimum value of  $y_j$ .

i.e.  $p' = (x', y', z)$  such that  $y' \leq y_j$ .

$$\text{Let } x_j = \frac{B_j y + C_j z + D_j}{-A_j} \text{ for } A_j \neq 0$$

Let  $x^+ = \min_{H_j \in H^c} (x_j)$  if  $H^c$  is not empty,  $x^+ = M-1$  if  $H^c$  is empty.

Let  $x^- = \max_{H_j \in H^w} (x_j)$  if  $H^w$  is not empty,  $x^- = -M+1$  if  $H^w$  is empty.

Letting  $H'''$  be the set  $\{H_j (A_j, B_j, C_j, D_j), H_j \in C, C_j < 0\}$  (i.e the subset of the half planes defining C for which  $C_j < 0$ ). Repeating the above logic, using  $z'$  instead of  $x'$ , leads to the conclusion that  $H = H''$ .

Clearly,  $H''$  is redundant to any convex polytope.

This means that any space  $H$ , such that all points  $p = (x, y, z) \in C^c$  satisfy  $Ax + By + Cz + D \geq 0$  is redundant to  $C$ .

The importance of this lemma is that in the implementation of an algorithm to simplify the complex polygons, it is sufficient to test that all vertex points of a bounded convex polygon are within the pseudo closure of a half plane to declare that half plane redundant. Various algorithms (such as the conversion of a general polygon to a regular polytope) result in cases similar to Figure 5, which would otherwise require tedious special case testing.



## 5 Vertex Test for Redundant Half Spaces

This result is very useful in implementing a database structure based on the regular polytope concept, since it allows the code which removes redundant half spaces or planes to be made much simpler. The testing of many special cases can be avoided.

### Proposition 1:

If  $H$  is a halver<sup>4</sup> such that  $C^{pc} \subseteq H^{pc}$ , then  $C \subseteq H$ .

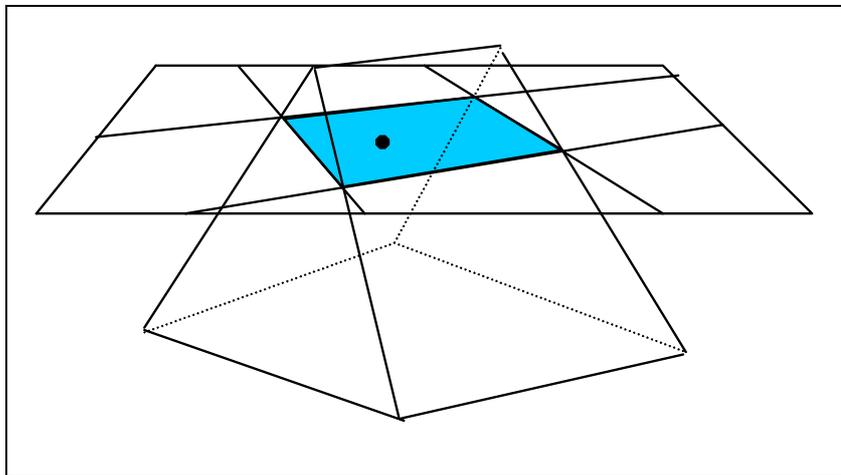
That is to say that if every point in  $C^{pc}$  is also within  $H^{pc}$ , then every point in  $C$  must also be within  $H$ , (and therefore  $H$  is redundant to the definition of  $C$ ).

### 5.1 Outline of Proof

The first step is to show that the “footprint” of an  $n$  dimensional halver/convex polytope on a hyperplane is itself a valid  $n-1$  dimensional halver/convex polytope, provided that the hyperplane is of the form  $x_n = c$  where  $x_n$  is the last<sup>5</sup> coordinate.

The remainder of the proof is by induction. Assuming the proposition is true for  $n-1$  dimensions, it is shown to hold for  $n$  dimensions.

Finally, the proposition is shown to hold for 1 dimension.



**Figure 6** The  $n-1$  dimensional object defined by the intersection of a convex polytope with a hyperplane  $x_n=c$ , is itself a convex polytope containing any point from the original polytope with  $x_n=c$ .

A changed notation is required for this proof to allow for variable dimensionality.

Let  $H(A_1, A_2 \dots A_n, B)$  be a halver in  $n$  dimensions defined as:

$$x = (x_1, x_2, \dots, x_n) \in H(A_1, A_2 \dots A_n, B) \text{ if}$$

<sup>4</sup> In this proof, the term “halver” is used to mean half plane, half space etc, depending on the dimension.

<sup>5</sup> “Last” in this sense is the last coordinate to be mentioned in the definition of  $p$ .

$$A_1x_1 + A_2x_2 + \dots + A_nx_n + B \geq 0 \text{ -- where}$$

$$\rho \text{ is } \geq \text{ if } A_1 > 0,$$

$$\rho \text{ is } > \text{ if } A_1 < 0,$$

$$\rho \text{ is } \geq \text{ if } A_1 = 0, A_2 > 0,$$

$$\rho \text{ is } > \text{ if } A_1 = 0, A_2 < 0,$$

$$\text{etc.}$$

$$\rho \text{ is } \geq \text{ if } A_1 \dots A_{n-1} = 0, A_n > 0$$

$$\rho \text{ is } > \text{ if } A_1 \dots A_{n-1} = 0, A_n \leq 0$$

An n dimensional halver is valid if:

$$A_1 \dots A_n \text{ and } B \text{ are integers,}$$

$$-M < A_1 \dots A_n < M,$$

$$-nM^2 < B < nM^2,$$

$$\text{and if } A_1 = A_2 = \dots = A_n = 0, \text{ then } B \neq 0.$$

$H^{pc}$  is defined as  $x \in H^{pc}$  if

$$A_1x_1 + A_2x_2 + \dots + A_nx_n + B \geq 0$$

Let  $C = \{H_1 \dots H_m\}$  be a convex polytope defined by m halvers defined as follows:

$$H_1 = H(A_{11}, A_{12}, \dots, A_{1n}, B_1)$$

$$H_2 = H(A_{21}, A_{22}, \dots, A_{2n}, B_2)$$

....

$$H_m = H(A_{m1}, A_{m2}, \dots, A_{mn}, B_m)$$

so that  $H_j$  is defined as  $\sum_{i=1..n} A_{ji}x_i + B_j \geq 0$

$C$  is defined as  $\{x: x \in H_j, j=1..m\}$ .

$C^{pc}$  is defined as  $\{x: x \in H_j^{pc}, j=1..m\}$ .

## 5.2 Lemma

The intersection of n-dimensional halver ( $n > 1$ ),  $H(A_1, A_2 \dots A_n, B)$  with the hyperplane  $x_n = x'_n$  (where  $x'_n$  is a constant  $-M \leq x'_n < M$ ) is itself a n-1 dimensional halver  $H$ . Further, any point  $p \in H$  is an element of  $H$ , and any point  $p \in H$  with  $x_n = x'_n$  is an element of  $H$ .

$$\text{Let } B'' = A_n x'_n + B,$$

The halver  $H(A_1, A_2 \dots A_{n-1}, B'')$  would fulfil the requirements of a valid halver if:

$$A_1 \dots A_{n-1} \text{ and } B'' \text{ are integers,}$$

$$-M < A_1 \dots A_{n-1} < M,$$

$$-nM^{n-1} < B'' < nM^{n-1},$$

$$\text{and if } A_1 = A_2 = \dots = A_{n-1} = 0, \text{ then } B'' \neq 0.$$

While the first two requirements are obvious, the latter two are not necessarily true, but with the following modifications, a valid halver can be assured.

$$\text{If } B'' \leq -(n-1)M^2, \text{ then let } B' = -(n-1)M^2 + 1. \quad ^6$$

$$\text{If } B'' \geq (n-1)M^2, \text{ then let } B' = (n-1)M^2 - 1. \quad ^7$$

$$\text{If } A_1 = A_2 = \dots = A_{n-1} = 0, \text{ and } B'' = 0, \text{ then}$$

<sup>6</sup> In fact, this is equal to the empty halver, since the sum of the  $A_i x_i$  terms can never be  $> (n-1)M^2 + 1$ .

<sup>7</sup> Similarly, this is equal to the universal halver.

let  $B'' = 1$  if  $A_n > 0$ ,  
let  $B'' = -1$  if  $A_n \leq 0$ .  
Otherwise,  $B' = B''$ .

It is clear that this is now a valid  $n-1$  dimension halver.

let  $x^- = (x_1, x_2, \dots, x_{n-1})$  be a  $n-1$  dimensional point  $x^- \in H$   
and let  $x^+ = (x_1, x_2, \dots, x_{n-1}, x'_n)$  be that point extended to  $n$  dimensions

Case 1  $A_1 \dots A_{n-1}$  are not all zero,  $B' = B''$ ,

$x^- \in H$

$\Rightarrow \sum_{i=1..n-1} A_i x_i + B' \geq 0$  (note – that  $\rho$  has the same definition as in the  $n$  dimensional halver)

$\Rightarrow \sum_{i=1..n-1} A_i x_i + A_n x'_n + B' \geq 0$

$\Rightarrow x^+ \in H$ .

Case 2  $A_1 \dots A_{n-1}$  are not all zero,  $B' = -(n-1)M^2 + 1$ .

by definition above,  $B'' \leq -(n-1)M^2 + 1$ .

since each  $A_i \leq M-1$ , and each  $x_i \leq M$ ,

$\sum_{i=1..n-1} A_i x_i \leq (n-1)M(M-1)$ , i.e.

$\sum_{i=1..n-1} A_i x_i \leq (n-1)M^2 - (n-1)M$

since  $n > 1$ , and  $M > 1$ ,  $\sum_{i=1..n-1} A_i x_i < (n-1)M^2 - 1$

$\sum_{i=1..n-1} A_i x_i + B' < 0$

thus, there is no point  $x^-$  in  $H$ .

Case 3  $A_1 \dots A_{n-1}$  are not all zero,  $B' = (n-1)M^2 - 1$ .

by definition above,  $B'' \geq (n-1)M^2$ .

consider  $x^+$  as defined above (such that  $x^- \in H$ ).

$\sum_{i=1..n-1} A_i x_i + A_n x'_n + B' = \sum_{i=1..n-1} A_i x_i + B''$ .

by similar reasoning to above,  $\sum_{i=1..n-1} A_i x_i > -(n-1)M^2 + 1$

therefore  $\sum_{i=1..n-1} A_i x_i + B'' > 0$

therefore  $x^+ \in H$ .

Case 4 If  $A_1 = A_2 = \dots = A_{n-1} = 0$ , and  $B' > 0$

By definition of  $B'$ ,  $A_n x'_n + B' > 0$  or  $A_n x'_n + B' = 0$  and  $A_n > 0$ .

In either case,  $x^+ \in H$ .

Case 5 If  $A_1 = A_2 = \dots = A_{n-1} = 0$ , and  $B' < 0$

$H$  is empty.

So in all cases,  $x^- \in H \Rightarrow x^+ \in H$ .

**Conversely**

Again let  $x^- = (x_1, x_2, \dots, x_{n-1})$  be a  $n-1$  dimensional point  $x^- \notin H$  and let  $x^+ = (x_1, x_2, \dots, x_{n-1}, x'_n)$  be that point extended to  $n$  dimensions

Case 1  $A_1 \dots A_{n-1}$  are not all zero,  $B' = A_n x'_n + B$ ,  
 $x^- \notin H$

$$\Rightarrow \sum_{i=1..n-1} A_i x_i + B' \bar{\rho} = 0 \quad (\text{where } \bar{\rho} \text{ is the inverse of } \rho)$$

$$\Rightarrow \sum_{i=1..n-1} A_i x_i + A_n x'_n + B \bar{\rho} = 0$$

$$\Rightarrow x^+ \notin H.$$

Case 2  $A_1 \dots A_{n-1}$  are not all zero,  $B' = -(n-1)M^2 + 1$ .

by definition above,  $B'' \leq -(n-1)M^2 + 1$ .

consider  $x^+$  as defined above (such that  $x^- \notin H$ ).

as above,  $\sum_{i=1..n-1} A_i x_i < (n-1)M^2 - 1$

therefore  $\sum_{i=1..n-1} A_i x_i + B'' < 0$

therefore  $x^+ \notin H$ .

Case 3  $A_1 \dots A_{n-1}$  are not all zero,  $B' = (n-1)M^2 - 1$ .

as above,  $\sum_{i=1..n-1} A_i x_i > -(n-1)M^2 + 1$

$$\sum_{i=1..n-1} A_i x_i + B' > 0$$

Thus every point  $x^-$  must be in  $H$ .

Case 4 If  $A_1 = A_2 = \dots = A_{n-1} = 0$ , and  $B' > 0$

no point  $x^-$  can exist such that  $x^- \notin H$ .

Case 5 If  $A_1 = A_2 = \dots = A_{n-1} = 0$ , and  $B' < 0$

By definition of  $B'$ ,  $A_n x'_n + B < 0$  or  $A_n x'_n + B = 0$  and  $A_n \leq 0$ .

In either case,  $x^+ \notin H$ .

So in all cases,  $x^- \notin H \Rightarrow x^+ \notin H$ .

Alternatively, any point  $p \in H$  with  $x_n = x'_n$  is an element of  $H$ .

**Corollary:**

If  $C$  is a convex polytope  $C = \{H_1 \dots H_m\}$ , and the intersection  $H_j^-$  of each  $H_j$  ( $j=1..m$ ) with the plane  $x_n = x'_n$  is formed as above, the set  $C = \{H_1^- \dots H_m^-\}$  is a  $n-1$  dimensional convex polytope. Further, for any point  $x \in C \Rightarrow x \in C$ , and for any point with  $x_n = x'_n$ ,  $x \in C \Rightarrow x \in C$ .

This follows from the definitions.

**Corollary:**

The intersection of the pseudo closure of a n-dimensional halver ( $n > 1$ ),  $H^{pc}$  with the hyperplane  $x_n = x'_n$  (where  $x'_n$  is a constant  $-M+1 \leq x'_n < M-1$ ) is itself a n-1 dimensional halver pseudo closure  $H^{pc}$ . Further, any point  $p \in H^{pc} \Rightarrow p \in H^{pc}$ , and for any point with  $x_n = x'_n$ ,  $p \in H^{pc} \Rightarrow p \in H^{pc}$ .

This follows using the logic of the above lemma.

**Corollary:**

If  $C^{pc}$  is a the pseudo closure of a convex polytope  $C^{pc} = \{H_1^{pc} \dots H_m^{pc}\}$ , and the intersection  $H_j^{pc}$  of each  $H_j^{pc}$  ( $j=1..m$ ) with the plane  $x_n = x'_n$  is formed as above, the set  $C^{pc} = \{H_1^{pc} \dots H_m^{pc}\}$  is the pseudo closure of an n-1 dimensional convex polytope. Further, any point  $x \in C^{pc} \Rightarrow x \in C^{pc}$ , and for any point with  $x_n = x'_n$ ,  $x \in C^{pc} \Rightarrow x \in C^{pc}$ .

This follows from the definitions.

**5.3 Proof of Proposition**

Let  $H$  be a halver such that every point in  $C^{pc}$  is within  $H^{pc}$ .

Assume that the proposition is true for dimension n-1. (The iterative assumption)

Let  $x = (x_1, x_2, \dots, x_n)$  be any point which is within  $C$ .

$$x \in C \Rightarrow x \in C^{pc} \Rightarrow x \in H^{pc}.$$

Consider the hyperplane defined by  $x$  (all points with the same value of  $x_n$ ), and the intersection of  $C$  and  $H$  with this hyperplane (named  $C$  and  $H$  respectively):

Let  $x^- = (x^-_1, x^-_2, \dots, x^-_{n-1})$  be any point in  $C^{pc}$ , then it follows that  $x^+ = (x^-_1, x^-_2, \dots, x^-_{n-1}, x_n) \in C^{pc}$ ,

$$x^+ \in C^{pc} \Rightarrow x^+ \in H^{pc}$$

$$x^+ \in H^{pc} \Rightarrow x^- \in H^{pc}$$

Thus any point in  $C^{pc}$  is also in  $H^{pc}$ . Therefore, by the iterative assumption,  $H$  is redundant to the definition of  $C$ , i.e.  $x^- \in C \Rightarrow x^- \in H$ .

But  $x \in C \Rightarrow x^- \in C \Rightarrow x^- \in H$ .

$$x^- \in H \Rightarrow x \in H.$$

Thus  $x \in C \Rightarrow x \in H$ .

i.e. if the proposition is true in n-1 dimensions, then it is true in n dimensions.

**5.4 Consider the 1D Case**

Let  $H(A, B)$  be a halver defined as:

$$x \in H(A, B) \text{ if}$$

$$Ax + B \geq 0 \text{ -- where}$$

$\rho$  is  $\geq$  if  $A > 0$ ,  
 $\rho$  is  $>$  if  $A < 0$ ,

Let  $C = \{H_1, \dots, H_m\}$  be a convex polytope defined by  $m$  halvers defined as follows:

$$H_1 = H(A_1, B_1)$$

$$H_2 = H(A_2, B_2)$$

....

$$H_m = H(A_m, B_m)$$

so that  $H_j$  is defined as  $A_j x + B_j \rho_j = 0$ .

Let  $\mathbf{H}$  be the set  $\{H_i: A_i > 0\}$ , and  $\mathbf{H}^+$  be the set  $\{H_i: A_i < 0\}$

If  $\mathbf{H}$  is non empty:

let  $x^-$  be the largest value of  $-B_i/A_i$  for any halver in  $\mathbf{H}$ .

consider  $H_i \in \mathbf{H}$  such that  $x^- = -B_i/A_i$ ,

if  $(x < x^-)$ , then  $A_i x + B_i < A_i x^- + B_i = A_i(-B_i/A_i) + B_i = 0$

i.e.  $x \notin C$

If  $\mathbf{H}$  is empty:

let  $x^- = -M$ .

for all points,  $x \geq -M$ .

If  $\mathbf{H}^+$  is non empty:

Let  $x^+$  be the smallest value of  $-B_i/A_i$  for any halver in  $\mathbf{H}^+$ .

consider  $H_i \in \mathbf{H}^+$  such that  $x^+ = -B_i/A_i$ ,

if  $(x \geq x^+)$ , then  $A_i x + B_i \leq A_i x^+ + B_i = A_i(-B_i/A_i) + B_i = 0$

i.e.  $x \notin C$

If  $\mathbf{H}^+$  is empty:

let  $x^+ = M$ .

for all points,  $x < M$ .

See Figure 7 for illustration of  $\mathbf{H}$  and  $\mathbf{H}^+$  in relation to the definition of the points which fall within  $C$ .

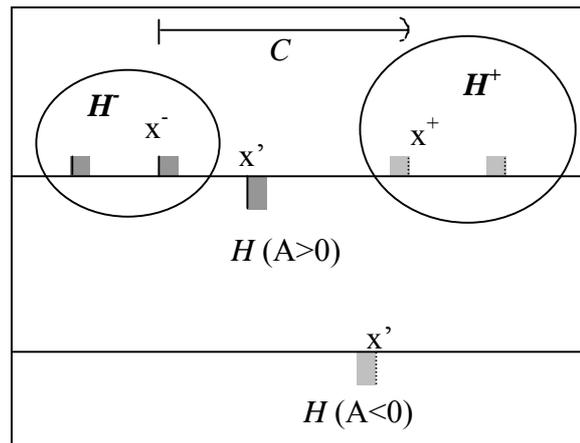


Figure 7 Convex polytope in 1D.

Therefore the convex  $C$  consists of those points which satisfy

$$x^- \leq x < x^+.$$

Similarly  $C^{pc}$  consists of those points which satisfy

$$x^- \leq x \leq x^+.$$

It is assumed that  $x \in C^{pc} \Rightarrow x \in H^{pc}$

If  $A > 0$ :

let  $x' = -B/A$ .

$H$  is defined as all points for which  $x \geq x'$  – and therefore  $H = H^{pc}$ .

$x^- \in C^{pc} \Rightarrow x^- \in H^{pc} \Rightarrow x^- \geq x'$ .

if  $x' \leq x^-$  all points in  $H$  are also in  $C$  – as required.

If  $A < 0$

let  $x' = -B/A$ .

$H$  is defined as all points for which  $x < x'$ .

$x^+ \in C^{pc} \Rightarrow x^+ \in H^{pc} \Rightarrow x^+ \leq x'$ .

if  $x' \geq x^+$  all points in  $H$  are also in  $C$  – as required.

If  $A = 0$

If  $B < 0$ , then  $H$  is the empty halver, and there are no points in  $H^{pc}$ . Thus  $C^{pc}$  is also empty.

Alternatively if  $B > 0$ , then this is the universal halver, and all points in  $C$  must be in  $H$ .

Thus the proposition is proved in 1 dimension.

Therefore it follows for all dimensions  $\geq 1$ .



## 6 Vertex Test for Incompatible Half Spaces

Taken in conjunction with proposition 1, this, in effect states that the complexity of the definition (in terms of the use of the  $\geq$  or  $>$  relational tests) can be safely ignored while programming the simplification algorithms. The complexity of these two proofs is more than justified by the reduction in complexity they allow in the software.

### Proposition 2:

$H^{px}$  (the pseudo-exterior of  $H$ ) is defined as:

$$x \in H^{px} \text{ if } A_1x_1 + A_2x_2 + \dots + A_nx_n + B \leq 0$$

note that any point on the boundary of  $H$  is an element of  $H^{px}$  and of  $H^{pc}$ .

If  $H$  is a halver<sup>8</sup> such that  $C^{pc} \subseteq H^{px}$ , then  $H$  is incompatible with  $C$ .

That is to say that if every point in  $C^{pc}$  is also within  $H^{px}$ , then every point in  $C$  is not within  $H$

### 6.1 Outline of Proof

The proof is by induction. Assuming the proposition is true for  $n-1$  dimensions, it is shown to hold for  $n$  dimensions.

Finally, the proposition is shown to hold for 1 dimension.

Let  $C = \{H_1 \dots H_m\}$  be a convex polytope defined by  $m$  halvers defined as follows:

$$H_1 = H(A_{11}, A_{12}, \dots, A_{1n}, B_1)$$

$$H_2 = H(A_{21}, A_{22}, \dots, A_{2n}, B_2)$$

....

$$H_m = H(A_{m1}, A_{m2}, \dots, A_{mn}, B_m)$$

so that  $H_j$  is defined as  $\sum_{i=1..n} A_{ji}x_i + B_j \leq 0$

$C$  is defined as  $\{x: x \in H_j, j=1..m\}$ .

$C^{pc}$  is defined as  $\{x: x \in H_j^{pc}, j=1..m\}$ .

### 6.2 Corollary

The intersection of the pseudo exterior of a  $n$ -dimensional halver ( $n > 1$ ),  $H^{px}$  with the hyperplane  $x_n = x'_n$  (where  $x'_n$  is a constant  $-M+1 \leq x'_n < M-1$ ) is itself a  $n-1$  dimensional halver pseudo exterior  $H^{px}$ . Further, any point  $p \in H^{px} \Rightarrow p \in H^{px}$ , and for any point with  $x_n = x'_n$ ,  $p \in H^{px} \Rightarrow p \in H^{px}$ .

This follows from the lemma of proposition 1.

---

<sup>8</sup> In this proof, the term "halver" is used to mean half plane, half space etc, depending on the dimension.

### 6.3 Proof of proposition 2

Let  $H$  be a halver such that every point in  $C^{pe}$  is within  $H^{px}$ .

Assume that the proposition is true for dimension  $n-1$ . (The iterative assumption)

Let  $x = (x_1, x_2, \dots, x_n)$  be any point such that  $x \in C$

$$x \in C \Rightarrow x \in C^{pe} \Rightarrow x \in H^{px}$$

Consider the hyperplane defined by  $x$  (all points with the same value of  $x_n$ ), and the intersection of  $C$  and  $H$  with this hyperplane (named  $C$  and  $H$  respectively):

Let  $x^- = (x_1^-, x_2^-, \dots, x_{n-1}^-)$  be any point in  $C^{pe}$ , then it follows that  $x^+ = (x_1^-, x_2^-, \dots, x_{n-1}^-, x_n)$   $\in C^{pe}$ ,

$$x^+ \in C^{pe} \Rightarrow x^+ \in H^{px}$$

$$x^+ \in H^{px} \Rightarrow x^- \in H^{px}$$

Thus any point in  $C^{pe}$  is also in  $H^{px}$ . therefore, by the iterative assumption,  $H$  is incompatible with  $C$ . Therefore there can be no point in  $C$  which is in  $H$ .

But  $x \in C \Rightarrow x^- \in C \Rightarrow x^- \notin H$ .

$$x^- \notin H \Rightarrow x \notin H.$$

Thus  $x \in C \Rightarrow x \notin H$ .

i.e. if the proposition is true in  $n-1$  dimensions, then it is true in  $n$  dimensions.

### 6.4 Consider the 1D Case

Defining  $H = H(A,B)$ , and  $C = \{H_1 \dots H_m\}$  as in proposition 1,

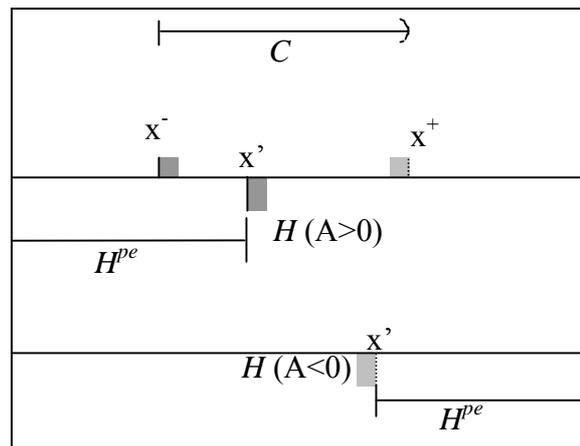


Figure 8 Relationship of  $x^-$ ,  $x^+$  and  $x'$  to  $C$  and  $H$

As before, we can determine  $x^-$  and  $x^+$  such that  $C$  is defined as

$$x, \text{ such that } x^- \leq x < x^+.$$

Similarly  $C^{pe}$  becomes

$$x, \text{ such that } x^- \leq x \leq x^+.$$

It is assumed that  $x \in C^{pe} \Rightarrow x \in H^{px}$

If  $A > 0$ :

$$\text{let } x' = -B/A.$$

$H$  is defined as all points for which  $x \geq x'$ .  
 $H^{px}$  is defined as all points for which  $x \leq x'$ .  
 $x^+ \in C^{pc} \Rightarrow x^+ \in H^{px} \Rightarrow x^+ \leq x'$ .  
 But  $x \in H \Rightarrow x \geq x' \Rightarrow x \geq x^+ \Rightarrow x \notin C$ .  
 i.e. all points in  $H$  are excluded from  $C$ .

If  $A < 0$

let  $x' = -B/A$ .  
 $H$  is defined as all points for which  $x < x'$   
 $H^{px}$  is defined as all points for which  $x \geq x'$ .  
 $x^- \in C^{pc} \Rightarrow x^- \in H^{px} \Rightarrow x^- \geq x'$ .  
 But  $x \in H \Rightarrow x < x' \Rightarrow x < x^- \Rightarrow x \notin C$ .  
 i.e. all points in  $H$  are excluded from  $C$ .

If  $A = 0$

If  $B < 0$ , then  $H$  is the empty halver, so no point in  $C$  may be in  $H$ .  
 Alternatively if  $B > 0$ , then  $H^{px}$  contains no points, so  $C$  must contain no points.

Thus the proposition is proved in 1 dimension.

Therefore it follows for all dimensions  $\geq 1$ .



## 7 Relationship between Convex Polytope Equality and Point Set Equality

### 7.1 Lemma. A Plane that Divides a Convex Polytope does so at a Non-degenerate Face

Let  $C = \{H_i : H_i, i=1..n\}$ , be a convex polytope, and consider the plane of dr-rational points  $H(A,B,C,D)$  defined by that  $p(x,y,z) \in H$  if  $Ax+By+Cz+D=0$ .

Let  $H'$  be the set of points such that  $p \in H, p \in C^{\text{PC}}$ .

If  $H'$  defines a degenerate face (co-linear points, a single point, or no points), then it is clear that, since CPC is a convex polyhedron, that all vertices lie on the same side of, or on the face  $H$ .

Now considering the half spaces  $H$  and  $\bar{H}$ , it is clear that this means that either  $H$  is redundant to the definition of  $C$ , and  $\bar{H}$  is incompatible to it, or visa-versa.

Thus either  $C \cap H = C_{\Phi}$ , or  $C \cap \bar{H} = C_{\Phi}$ . That is to say, the plane  $H$  does not divide the convex polytope  $C$  into two non-empty parts.

### 7.2 Definition of Convex Polytope Equality

Two convex polytopes,  $C = \{H_i : H_i, i=1..n\}$ , and  $C' = \{H'_j : H'_j, j=1..m\}$  are defined as being strongly equal ( $C \cong C'$ ) if there exists a 1-1 relationship between the half spaces of each such that  $H_i=H'_j$  for some pairs of  $i$  and  $j$ . Clearly this is only possible if  $n=m$ . Assume, for simplicity that  $C$  or  $C'$  has been reordered so that  $H_i=H'_i$  for  $i = 1..n$ . (The requirement that  $H=C'$  is the above definition of half space equality).

A convex polytope may contain half spaces which are redundant, in that their removal from the definition of the convex polygon does not affect the points defined as being within the convex polygon.

I.e if  $C = \{H_i : H_i, i=1..n\}$ , and  $C' = \{H_i : H_i, i=1..n-1\}$  then if point  $p \in C \Leftrightarrow p \in C'$ , then half space  $H_n$  is redundant. This can be relaxed to the requirement that  $p \in C' \Rightarrow p \in C$ , since the reverse must be true.

That is to say that if all the points within  $C' = \{H_i : H_i, i=1..n-1\}$  lie within the half plane  $H_n$ , then  $H_n$  is redundant. In practice, as will be shown in section 4, if all the pseudo-rational vertices that can be calculated for  $C'$  are within  $H_n$ , then  $H_n$  is redundant.

A convex polytope may also contain an incompatible half space, That is to say that if all the points within  $C' = \{H_i : H_i, i=1..n-1\}$  lie outside the half space  $H_n$ , then  $H_n$  is incompatible with the remaining half spaces. The presence of any incompatible half space means that the convex polytope is empty.

**Simplification** of a convex polytope is defined as the removal of any redundant half spaces, and if any incompatible half spaces are found, conversion of the convex polytope to the empty convex polytope  $C_{\Phi}$ .

Two convex polygons are equal if, after simplification, the resultant convex polygons are strongly equal. i.e.  $C^3 = C^4$  if  $C^1 \cong C^2$ , where  $C^1$  is the simplification of  $C^3$ , and  $C^2$  is the simplification of  $C^4$ .

### 7.3 Proof: Equal Convex Polygons are Equal in a Point-set Sense

Let  $C = \{H_i : H_p, i=1..n_1\}$ , and  $C' = \{H'_j : H'_p, j=1..n_2\}$  be two simplified convex polytopes such that  $C \cong C'$ .

Without loss of generality, assume that the half spaces have been re-ordered so that  $C = \{H_i : H_p, i=1..n\}$ , and  $C' = \{H'_i : H'_p, i=1..n\}$  and  $H_i=H'_i$  for  $i = 1..n$ .

Let  $p \in C$ ,

By definition,  $p \in H_i$  for  $i = 1..n$ .

Since  $H_i=H'_i$  for  $i = 1..n$ .

It follows that  $p \in H'_i$  for  $i = 1..n$ . (By the equivalence of point set and half-space equality).

Therefore  $p \in C'$ .

So  $p \in C \Rightarrow p \in C'$ .

Equivalently,  $p \in C' \Rightarrow p \in C$ .

So that  $C \cong C' \Rightarrow (p \in C \Leftrightarrow p \in C')$ .

### 7.4 Proof of Converse

#### 7.4.1 Outline of Proof

The converse argument will be proved by induction on the number of half spaces in the convex polytopes. Thus, it is assumed to be true of convex polytopes containing  $r$  half planes. This is then shown to imply that it is true for convex polytopes containing  $r+1$  half planes. Since it is true of convex polytopes consisting of one half plane, it must be true for all convex polytopes.

#### 7.4.2 Proof

Assume that this is true for all convex polytopes of with  $r$  non-redundant half planes in their definition, that is to say that for  $C^1 = \{H_i : H_p, i=1..r\}$ , and  $C^2 = \{H'_i : H'_p, j=1..r\}$ , if  $p \in C^1 \Leftrightarrow p \in C^2$ , then there exists a re-ordering of  $C^1$  or  $C^2$  such that  $H_i=H'_i$  for  $i = 1..r$ .

Consider  $C^3 = C^1 \cap H_{r+1}$  and  $C^4 = C^2 \cap H'_{r+1}$ . such that  $p \in C^3 \Leftrightarrow p \in C^4$ .

#### 7.4.3 Case 1 Redundant Half Spaces

If  $H_{r+1}$  is redundant to  $C^1$ , i.e.  $p \in C^1 \Leftrightarrow p \in C^3$ ,

$p \in C^1 \Leftrightarrow p \in C^2$  by the initial assumption.

$p \in C^3 \Leftrightarrow p \in C^4$  by definition of  $C^3$  and  $C^4$ .

Therefore  $p \in C^2 \Leftrightarrow p \in C^4$

Thus  $H'_{r+1}$  is redundant to  $C^4$ .

By the same reasoning  $H'_{r+1}$  is redundant to  $C^4 \Rightarrow H_{r+1}$  is redundant to  $C^3$ .

Therefore  $C^3 \cong C^4$ .

#### 7.4.4 Case 2 Empty Convex Polygons

If  $C^3 = \{H_i : H_i, i=1..r+1\}$  is empty, i.e.  $\forall p, p \notin C^3$ , this means that  $\forall p, p \notin C^4$ , therefore  $C^4$  is empty.

Since simplification of any empty convex polytope yields  $C_\emptyset$ ,  
 $C^3 = C^4$ .

#### 7.4.5 Case 3 $C1 \neq C3$ , and $C3$ not Empty

Consider the set of points  $p(x,y,z)$  such that  $Ax+By+Cz+D=0, p \in C^4$ . That is the set of pseudo-rational points on the plane which defines  $H_{r+1}$  that fall within  $C^4$ .

If these points are co-linear, or a single point, it follows by the above lemma (see 7.1) that  $H_{r+1}$  is either incompatible with (see case 2 above) or redundant to  $C^4$  (see case 1).

If the points are not co-linear, then it follows, by the reasoning of 3.4 that  $H_{r+1} = H'_{r+1}$ .  
Therefore  $C^3 = C^4$ .



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