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A system of types and operators for handling vague spatial objects

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Vagueness is often present in spatial phenomena. Representing and analysing vague spatial phenomena requires vague objects and operators, whereas current GIS and spatial databases can only handle crisp objects. This paper provides mathematical definitions for vague object types and operators.

The object types that we propose are a set of simple types, a set of general types, and vague partitions. The simple types represent identifiable objects of a simple structure, i.e. not divisible into components. They are vague points, vague lines, and vague regions. The general types represent classes of simple type objects. They are vague multipoint, vague multiline, and vague multiregion. General types assure closure under set operators. Simple and general types are defined as fuzzy sets in $\mathbb{R}^2$ satisfying specific properties that are expressed in terms of topological notions. These properties assure that set membership values change mostly gradually, allowing stepwise jumps. The type vague partition is a collection of vague multiregions that might intersect each other only at their transition boundaries. It allows for a soft classification of space. All types allow for both a finite and an infinite number of transition levels. They include crisp objects as special cases.

We consider a standard set of operators on crisp objects and define them for vague objects. We provide definitions for operators returning spatial types. They are regularized fuzzy set operators: union, intersection, and difference; two operators from topology: boundary and frontier; and two operators on vague partitions: overlay and fusion. Other spatial operators, topological predicates and metric operators, are introduced giving their intuition and example definitions. All these operators include crisp operators as special cases. Types and operators provided in this paper form a model for a spatial data system that can handle vague information. The paper is illustrated with an application of vague objects in coastal erosion.

Keywords: Spatial data modelling; Vagueness; Fuzzy sets; Fuzzy topology

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1. Introduction

Vagueness is a type of imperfection arising in the presence of borderline cases (Sorensen 2003). A concept is vague if locations exist that cannot be classified either to the concept or to its complement. When mapping vegetation for example, it may be difficult to decide whether a certain location belongs to one vegetation class or to another. The transition from one class to another may be gradual (Fisher 2000), as between forest and grassland in African rangelands. Also, geomorphological units (Burrough *et al.* 2000, Lucieer *et al.* 2004), soil types (de Gruijter *et al.* 1997, Odeh *et al.* 1992), landscape objects (Cheng and Molenaar 1999, Cheng *et al.* 2001, Fisher *et al.* 2006), and forest types (Brown 1998), generally exhibit transition zones instead of sharp boundaries. Other examples include soil pollution classes in environmental applications (Hendricks Franssen *et al.* 1997), or hydrological studies (Bogàrdi *et al.* 1990) where spatial objects have to be delineated that cannot be sharply defined.

Several theories have been proposed to handle vagueness, of which fuzzy theory (Zadeh 1965, 1975) is the most often used. This is also true in the spatial domain. Attention has been given to vagueness, on the one hand by considering spatial objects to be crisp, and reasoning being vague (Vert *et al.* 2002, Wang 2000). On the other hand, models have been proposed for objects that are vague (Burrough and Frank 1996, Cohn and Gotts 1996b, Erwig and Schneider 1997, Zhan 1997, Schneider 1999, Roy and Stell 2001, Tang 2004) and reasoning with these objects (see section 4).

Some image processing software allows the extraction of fuzzy objects from imagery, or offers tools to do fuzzy reasoning over a continuous space (Robinson 2003). There is also work done for the visualization of fuzzy objects (Hengl *et al.* 2004). No functionality exists, however, for handling vague objects in current geographical information systems (GISs), or in spatial database systems. These systems focus on crisp spatial objects. Current GISs or spatial databases would therefore benefit from an extension with data types for vague spatial objects, and operators for their manipulation.

Before constructing an information system, a precise notion must exist of what is to be built and how the system is expected to function. Mathematical specifications are useful in this respect. They describe precisely the properties the system must have, without unduly constraining how these properties are achieved in its implementation. These specifications, called abstract specifications, are not oriented towards computer representation, but obey a rich collection of mathematical laws. As such one can reason effectively about the way the system will behave. The abstract specifications are later translated into computer specifications, which are related to a specific computer representation. The work presented here deals only with abstract specifications.

The objective of this paper is to provide a model for a spatial data system that can handle vague objects. This model takes gradual transitions into consideration and is based on fuzzy theory. It consists of mathematical definitions for vague spatial types and operators. Vague spatial types are grouped into simple types, general types, and vague partitions. A simple type represents an identifiable object of a simple structure, i.e. not divisible into components. A general type represents a class of simple type objects. A vague partition allows for a soft classification of space. Vague operators defined here are operators returning spatial types. Intuition and example definitions are provided for other spatial operators.

The paper is organized as follows. Section 2 presents examples of vague spatial phenomena that can be described with our vague types. Section 3 gives a short introduction to concepts from fuzzy topology that are needed for our definitions of vague types and operators, also from some of the existing models for vague data. Section 4 describes existing models proposed to handle spatial vagueness. Section 5 contains the core material of the paper. It provides intuition and definitions of vague object types and operators returning spatial types. Section 6 discusses what is a complete spatial system, and elaborates on the other spatial operators, spatial predicates and metric operators. An application from coastal erosion is presented in section 7 to illustrate the object types and operators provided in this paper. Section 8 closes the paper with discussions and conclusions.

2. Examples of spatial vagueness

In ordinary natural language adjectives are commonly attributed to phenomena. This is certainly the case in the cognition of geography, where characteristics of spatial phenomena are expressed in ordinary language terms, which are generally vague.

(i) When considering densely populated residential centres, we have to identify known locations that have different degrees of being densely populated. We know the location precisely, but the density level itself is vague.
(ii) Considering polluted rivers, we know precisely where the river is. Close to the source of pollution the river is certainly polluted, but further away riverine tracks may exist with less severe pollution. Therefore, at different locations along the river, different degrees of pollution exist that change gradually.
(iii) Traffic congestion on a road network relates the level of congestion to roads. Part of the road network is completely blocked, and hence certainly belongs to the traffic congestion, whereas away from the congestion, the car build-up becomes less severe. Dissolving congestion at the end of a rush hour no longer has a certain part. This vague characteristic, however, is still spread on the roads.
(iv) It may be arbitrary to consider a particular location as part of a vegetated area, or of a non-vegetated area. Some locations are certainly vegetated, whereas other locations can be considered vegetated or non-vegetated to some degree. There are transition zones where vegetation becomes sparse. In some places the transition is gradual, whereas in other places the change may be abrupt.
(v) Agricultural land suitability in Sicat *et al.* (2005) is based on farmers' knowledge on soils, like soil texture, colour, depth and slope. These parameters are linguistic variables taking values, e.g. 'fine', 'moderately fine', and 'coarse' for soil texture. The suitability map is built from a combination of values of these variables according to specific rules. Suitability derived in this way is also a linguistic variable with values 'least suitable', 'moderately suitable', 'suitable', and 'most suitable.' This attribute, suitability, spread over the space, determines objects whose locations have membership values from a finite set (of four values). Two adjacent land parcels could have different suitability values, therefore there is a jump in values along their common boundary.

The above examples illustrate thematic vagueness, but not always locational vagueness. Objects describing such spatial phenomena may have a crisp location, but their essential properties can only be expressed in vague terms. Regions resulting from a classification of space have a vague extent due to the vagueness of concepts defining the classes. For such regions, locational vagueness derives from thematic vagueness. There are situations, however, where locational vagueness is independent of thematic vagueness, as for example a forest region from which one way or another we know the shape, but not the precise location. The object types proposed in this paper can handle a thematic kind of vagueness.

3. Preliminaries

Definitions of vague spatial types and operators use concepts from general and fuzzy topologies. We assume that concepts from fuzzy set theory and general topology are known (see Klir and Yuan (1995) for a complete treatment of fuzzy sets, and Willard (1970) and Kelley (1975) for general topology). The general topologies that we use are the usual topologies T_1 for $\mathbb{R}$, and T_2 for $\mathbb{R}^2$. The fuzzy topologies used for our definitions are $\mathcal{T}_1$ induced by T_1, and $\mathcal{T}_2$ induced by T_2. Induced fuzzy topologies need the concept of semi-continuous functions. This is explained in the next few paragraphs, together with the relation to continuous functions. Section 3.1 introduces shortly concepts from fuzzy topology. Definitions are given for real spaces $\mathbb{R}^n$.

A function f:$\mathbb{R}^n \rightarrow \mathbb{R}$ is *lower semi-continuous* at p iff for all $c<f(p)$ there exists a neighbourhood U_p of p such that for all $q \in U_p$, $c<f(q)$. The function f is lower semi-continuous on R^n iff it is lower semi-continuous at every $p \in \mathbb{R}^n$ (Jost 1998). Figure 1(b) shows the graph of a lower semi-continuous function g:$\mathbb{R} \rightarrow \mathbb{R}$. The function is almost everywhere continuous, except for three points x_0, 0 and x_1 where it is lower semi-continuous. For a value $c<g(0)$, a neighbourhood U_0 is drawn, such that any point $q \in U_0$ has a value $g(q)$ greater than c.

Correspondingly, a function f:$\mathbb{R}^n \rightarrow \mathbb{R}$ is *upper semi-continuous* at p iff for all $c>f(p)$ there exists U_p such that for all $q \in U_p$, $c>f(q)$ (Jost 1998). The function $\chi_{[a,\, b]}$ in figure 1(c) is upper semi-continuous. It is the characteristic function of the closed interval $[a, b]$. The property is general: the characteristic function of a closed set in ($\mathbb{R}^n$, T_n) is an upper semi-continuous function, whereas the characteristic function of an open set is a lower semi-continuous function.

A function f:$\mathbb{R}^n \rightarrow \mathbb{R}$ is continuous at a point p iff it is both upper and lower semi-continuous at p. The function f is continuous on $\mathbb{R}^n$ iff it is both upper and lower semi-continuous on $\mathbb{R}^n$. Figure 1(a) shows the graph of a continuous function f:$\mathbb{R} \rightarrow \mathbb{R}$.

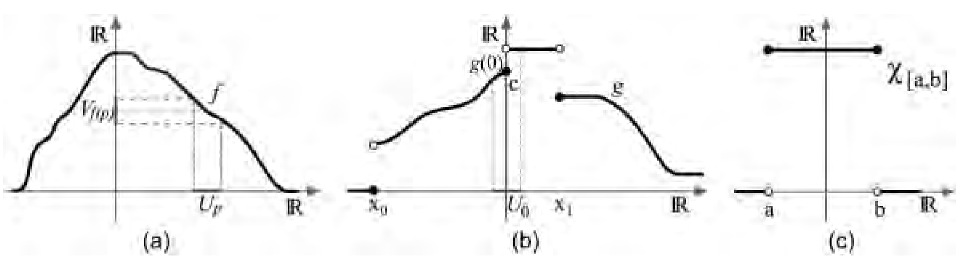

Figure 1. Graphs of continuous and semi-continuous functions for the usual topology in $\mathbb{R}^2$: (a) a continuous function, (b) a lower semi-continuous function and (c) an upper semi-continuous function. Empty and filled circles are used to show the value of a function at discontinuity points; the full circle denotes the value of the function.

A *fuzzy set* μ in $\mathbb{R}^n$ is a (total) function $\mu:\mathbb{R}^n \rightarrow [0, 1]$. This function is also called the membership function of the fuzzy set. A classical set A can be represented by its characteristic function $\chi_A:\mathbb{R}^n \rightarrow \{0, 1\}$, which is called a *crisp set*. We denote the set of all fuzzy sets in $\mathbb{R}^n$ by $\mathcal{F}(\mathbb{R}^n)$.

Let X and Y be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A function $f:X \rightarrow Y$ induces a function $\widetilde{f} : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ that produces an image of a fuzzy set in X as a fuzzy set in Y. This is known as the extension principle (Klir and Yuan 1995). The image of a fuzzy set $\mu \in \mathcal{F}(X)$ is $v = \widetilde{f}(\mu) \in \mathcal{F}(Y)$ such that

$$\forall p \in Y,\ v(p) = \begin{cases} \sup\{\mu(q) \mid q \in X, f(q) = p\}, & \exists q \in X, f(q) = p, \\ 0, & \text{otherwise.} \end{cases}$$

The fuzzy set v is such that its membership value at a location $p \in Y$ is the highest membership value of μ at locations $q \in X$ that are mapped to p by the function f.

3.1 *Fuzzy topology*

We now turn to definitions from fuzzy topology that are needed for our vague types: interior and closure, boundary, connected and bounded fuzzy sets. Most of these fuzzy topology notions are described by one of the equivalent definitions of the corresponding notion in general topology, put in a fuzzy setting. We give two examples of fuzzy topologies for real spaces, a crisp topology and an induced fuzzy topology. The topology notions are explained for the induced fuzzy topologies $\mathcal{T}_n$ for $\mathbb{R}^n$, and some of them are illustrated for the induced topology $\mathcal{T}_1$ for $\mathbb{R}$.

A fuzzy topology is defined similarly to a general topology as a collection $\mathcal{T} \subseteq \mathcal{F}(\mathbb{R}^n)$ that contains the empty set $0^{\mathbb{R}^n}$ and the whole set $1^{\mathbb{R}^n}$, the intersection of any two elements of $\mathcal{T}$, and any union of elements of $\mathcal{T}$. The elements of $\mathcal{T}$ are the open fuzzy sets in $(\mathbb{R}^n, \mathcal{T})$. Their complements are the closed fuzzy sets in $(\mathbb{R}^n, \mathcal{T})$.

The collection $\mathcal{C}_n$ of crisp sets from $\mathbb{R}^n$ which (corresponding classical sets) are open in the usual topology, forms a fuzzy topology for $\mathbb{R}^n$. Closed fuzzy sets for this topology are the closed crisp sets in the usual topology for $\mathbb{R}^n$. The fuzzy topology $\mathcal{C}_n$ is a crisp topology for $\mathbb{R}^n$. A general topology can be used differently to build a fuzzy topology. The collection $\mathcal{T}_n$ of lower semi-continuous functions in $(\mathbb{R}^n, T_n)$ forms a fuzzy topology, which is called the *induced fuzzy topology* from the usual topology for $\mathbb{R}^n$ (Weiss 1975). The fuzzy sets with lower semi-continuous membership function are the open sets in $\mathcal{T}_n$, and those with upper semi-continuous membership function are the closed sets. Open and closed fuzzy sets for $\mathcal{T}_n$ can be expressed in terms of open and closed α-cuts: a fuzzy set μ in $(\mathbb{R}^n, \mathcal{T}_n)$ is open if all its strict α-cuts are open for T_n; it is closed for $(\mathbb{R}^n, \mathcal{T}_n)$ if all its α-cuts are closed for T_n (Weiss 1975). A fuzzy set that has a continuous function is an open and closed (clopen) fuzzy set, e.g. the fuzzy set in $\mathbb{R}$ whose graph is shown in figure 1(a).

A *fuzzy point* in $(\mathbb{R}^n, \mathcal{T}_n)$ is a fuzzy set that has a positive value $\lambda > 0$ in just one point, say $p \in \mathbb{R}^n$ (Pu and Liu 1980):

$$\forall q \in \mathbb{R}^n,\ \mu(q) = \begin{cases} \lambda \text{ if } q = p, \\ 0 \text{ if } q \neq p. \end{cases}$$

A fuzzy point is denoted by p_λ, where p is the unique location with a positive membership, and λ is the membership value at p.

The *interior* of a fuzzy set μ is the union of all open sets contained in μ: $\mu^{\circ} = \sqcup\{v \in \mathcal{T} | v \sqsubseteq \mu\}$. The *closure* of a fuzzy set μ is the intersection of all closed sets containing μ: $\bar{\mu} = \sqcap\{1 - v \in \mathcal{T} | \mu \sqsubseteq v\}$. The interior puts the value of the function at the discontinuity points to the lower value. The closure sets the value of the function at discontinuity points to the highest value. A fuzzy set μ is *regular closed* iff it is equal to the closure of its interior: $\mu = \overline{\mu^{\circ}}$.

The *fuzzy boundary* μ^{b} of a fuzzy set μ is the intersection of all closed sets v in $\mathcal{F}(\mathbb{R}^n)$ such that $v(x) \geqslant \bar{\mu}(x)$ at all $x \in \mathbb{R}^n$ for which $\bar{\mu} \sqcap \overline{1 - \mu}(x) > 0$ (Warren 1977). The fuzzy boundary of a set μ in $(\mathbb{R}^n, \mathcal{T}_n)$ is equal to μ at the uncertain part $\{p \in \mathbb{R}^n | 0 < \mu(p) < 1\}$, has value 1 at the (crisp) boundary of the core, and value 0 everywhere else. The *fuzzy frontier* μ^{f} of μ is the intersection of all closed sets v in $\mathcal{F}(\mathrm{R}^n)$ such that $v(x) \geqslant \bar{\mu}(x)$ at all $x \in X$ for which $\bar{\mu}(x) > \mu^{\circ}(x)$ (Cuchillo-Ibáñez and Tarrés 1997). The frontier μ^{f} of a fuzzy set μ in $(\mathbb{R}^n, \mathcal{T}_n)$ has a positive value only at the discontinuity points of μ. It is an empty fuzzy set if μ is continuous.

Let μ be a fuzzy set in $(\mathbb{R}^n, \mathcal{T}_n)$, and $X \subset \mathbb{R}^n$. The fuzzy set $\mu_{|X}$ on $\mathbb{R}^n$ that has the same membership value as μ for all $x \in X$ and value 0 for all $x \notin X$ is a fuzzy set on X. We call the set $\mu_{|X}$ on $\mathbb{R}^n$ the restriction of μ on X. The family $\mathcal{T}_n^X = \left\{\mu_{|X} | \mu \in \mathcal{T}_n\right\}$ is a fuzzy topology in X and is called the relative fuzzy topology for X (Pu and Liu 1980).

A fuzzy set μ is disconnected if there are closed sets γ and δ in the subspace $supp(\mu)$ associated with the relative fuzzy topology, such that $\mu \sqcap \gamma \neq 0^{\mathbb{R}^n}$, $\mu \sqcap \delta \neq 0^{\mathbb{R}^n}$, $\gamma \sqcap \delta = 0^{\mathbb{R}^n}$, and $\mu \sqsubseteq \gamma \sqcup \delta$ (Pu and Liu 1980). A fuzzy set μ is *connected* if it is not disconnected. The maximal connected fuzzy set contained in μ is called a *component* of μ. A fuzzy set μ is connected for $(\mathbb{R}^n, \mathcal{T}_n)$ if it has a connected support set. A fuzzy set μ is bounded if every α-cut μ_{α} is bounded (Pu and Liu 1980). The set μ is bounded in $(\mathbb{R}^n, \mathcal{T}_n)$ if its support set is bounded.

4. Existing models of vague spatial data

Several theoretical models have been proposed, mainly dedicated to vague regions and their topological relations. These models can be grouped into two approaches. One approach (Burrough and Frank 1996, Cohn and Gotts 1996b, Erwig and Schneider 1997, Bittner and Stell 2000, Clementini and di Felice 2001, Roy and Stell 2001) considers vague regions to have a homogeneous two-dimensional boundary instead of a one-dimensional boundary. Locations in the broad boundary all have the same degree of membership to the region. Models of this approach do not provide means to handle gradual transition. The other approach (Zhan 1997, 1998, Schneider 1999, 2003, Tang 2004, Liu and Shi 2006) employs fuzzy set theory for modelling gradual changes. In particular, the work of Schneider (1999, 2000, 2001a) provides a complete set of object types and most of the standard operators of a spatial data system (Güting 1994). The next two sections summarize the works from each approach.

4.1 *Broad boundary regions*

The Egg-Yolk model (Cohn and Gotts 1996a,b) describes a vague region as a pair of crisp regions, one enclosing the other. The inner region, the 'yolk', gives the certain part of the vague region. The outer region, the 'white', is the broad boundary which delineates limits on the range of vagueness. The white and yolk together form the egg that is the full extent of the vague region. All these regions are Region

Connection Calculus (RCC) regions (Randell and Cohn 1989, Randell *et al.* 1992, Cohn *et al.* 1997).

The proposal of Clementini and di Felice (1996, 2001) is similar to the egg-yolk model, with definitions based on general topology. A vague region A consists of two sets A_1, A_2 of $\mathbb{R}^2$ such that $A_1 \subseteq A_2$. The broad boundary ΔA is the closure of their difference: $\Delta A = \overline{A_2 \backslash A_1}$. Each set, A_1 and A_2, is a crisp region, i.e. a bounded, regular closed set in $\mathbb{R}^2$ with connected interior (Clementini and di Felice 2001). The inner region A_1 gives the certain part of a vague region, and the broad boundary ΔA delineates limits of vagueness.

A vague region is defined in Erwig and Schneider (1997) as a pair of disjoint sets: the kernel that is the certain part of the region, and the boundary that is its uncertain part. The kernel is a crisp region, whereas the boundary can be a region or a line, the latter allowing a crisp region to be a special case of a vague region. This feature is not supported by the Egg-Yolk and Clementini and di Felice models.

Finally, rough sets are used to model vague regions in Bittner and Stell (2000) and Roy and Stell (2001). A rough set is represented by a pair of classical sets, called the lower and upper approximation. The lower approximation consists of all the elements that certainly belong to the set, whereas the upper approximation consists of all elements that possibly belong to the set (Pawlak 1994). A vague region is modelled in Roy and Stell (2001) by a pair of RCC regions representing the lower and upper approximation. When the lower and upper approximation are equal, the region is crisp. This model is a generalization of the Egg-Yolk model. Three-valued Łukasiewicz algebras are used as a formal context for vague regions and operators.

4.2 *Fuzzy spatial objects*

Mathematical definitions of fuzzy regions and topological relations between them are provided in Zhan (1997, 1998). Fuzzy regions are represented as fuzzy sets in Zhan (1997). This allows the existence of irregularities, e.g. isolated points or lines that are not desirable for regions. Fuzzy regions are redefined in Zhan (1998) in terms of fuzzy convexity. This excludes irregularities, but it is unnecessarily restrictive.

Schneider's fundamental work provides formal definitions of fuzzy types (Schneider 1999, 2003), and definitions of topological and metric operators on fuzzy objects (Schneider 2000, 2001a,b). Fuzzy points, fuzzy lines and fuzzy regions are defined as fuzzy sets from $\mathbb{R}^2$ in Schneider (1999). The definitions of fuzzy regions and their operators are built from a regularization function expressed as a combination of interior and closure operators, but without specifically indicating the employed topology. The regularization function applied to a fuzzy set gives different fuzzy sets for different fuzzy topologies. The function is thus ambiguous, which in turn leads to ambiguity in the definitions of region types and operators. In the next paragraph we show this ambiguity by an example.

Let $\psi \in \mathbb{R}^2$ be a fuzzy set defined by

$$\forall p \in \mathbb{R}^2,\ \psi(p) = \begin{cases} 1 & \text{for } \mathrm{d}_2(p, O) \leq 1, \\ 2 - \mathrm{d}_2(p, O) & \text{for } 1 < \mathrm{d}_2(p, O) \leq 2, \\ 0 & \text{for } \mathrm{d}_2(p, O) > 2, \end{cases}$$

where d_2 is the Euclidean distance in $\mathbb{R}^2$, and O is the origin (0, 0). Figure 2(a) illustrates the fuzzy set ψ using saturation to show membership values; the boundary

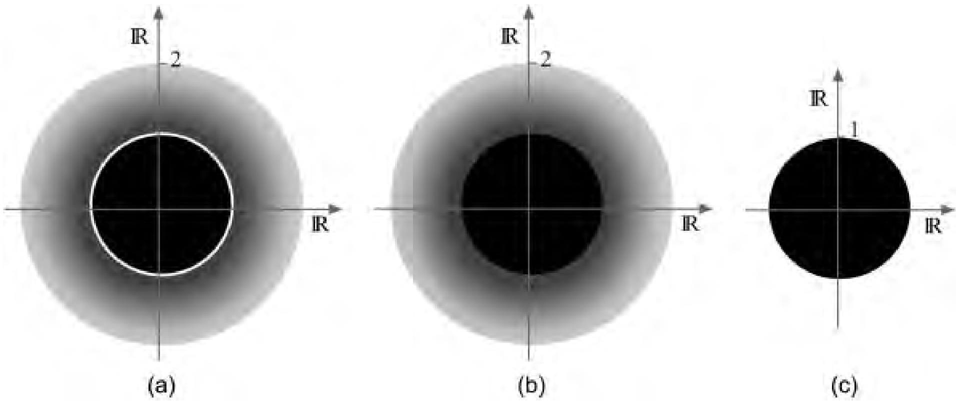

Figure 2. Regularization of a fuzzy set for two different fuzzy topologies: (*a*) a fuzzy set in $\mathbb{R}^2$ with the core boundary drawn in white, (*b*) its regularization for $\mathcal{T}_2$ and (*c*) its regularization for $\mathcal{C}_2$. Stronger tone indicates higher membership value, lighter tone indicates lower membership.

of the core is drawn in white. Let us consider two fuzzy topologies, the induced fuzzy topology from the usual topology $\mathcal{T}_2$, and the crisp topology $\mathcal{C}_2$ built from T_2. A frontier notion is used in Schneider (1999) to build the regularization function. The frontier of a fuzzy set is its restriction to the difference of its support set with the support set of its interior: $frontier(\psi)=\psi_{|supp(\psi)\setminus supp(\psi^\circ)}$. The regularization function is then defined as $reg(\psi)=\bar{\psi}^\circ \sqcup \left(frontier(\psi)\sqcap frontier\left(\overline{\psi^\circ}\right)\right)$. Application of the regularization function on ψ for the topology $\mathcal{T}_2$ gives the set itself: $reg_{\mathcal{T}_2}(\psi)=\psi$. Such regularization for the crisp topology gives the crisp closed unit disk: $reg_{\mathcal{C}_2}(\psi)=\chi_{\overline{U(O,\,1)}}$. Regularization yields different results when performed for different topologies. Therefore, introducing it without specifying a topology makes the definitions ambiguous.

Fuzzy objects are defined in Schneider (2003) based on a finite collection of elements from a regular grid, forming a partition of a bounded subspace of $\mathbb{R}^2$. Membership values are assigned to the elements of the grid: points, edges, and cells. Each fuzzy object is built from the grid elements. The model eliminates anomalies of calculations on real numbers performed with a finite set of numbers available in computers. On the other side, it is an abstract model that is built upon current possibilities of computer representations for spatial data.

Two definitions of fuzzy regions are provided in Tang (2004) using the crisp topology $\mathcal{C}_2$ and a fuzzy topology. The fuzzy regions are defined as fuzzy sets satisfying a list of properties in these topological spaces. The crisp topology produces a coarse representation: gradual transitions cannot be expressed by these definitions. The fuzzy topology is not specified, which makes the definitions prone to ambiguity, in the same way ambiguity arises in Schneider (1999).

5. Vague spatial types and operators returning spatial types

This section provides formal definitions of vague spatial types and of operators returning these types. We distinguish between simple types and general types. A simple type represents an identifiable object with the simplest structure, i.e. non-divisible into components. The simple types are VPoint, VLine and VRegion, representing a vague point, a vague line, and a vague region, respectively. A general type represents a class of simple objects. The general types are VMPoint, VMLine and VMRegion, representing a vague multipoint, a vague multiline, and a vague multiregion, respectively. The operators are regularized set operators: union, intersection and difference, together with two operators from topology: frontier and boundary. The general types assure closure for the set operators, and the frontier operator. The return type of a boundary operator is none of the above types. To cover for this, we propose two other types, VExt and VLDim. A VExt object is a collection of vague lines and vague regions, a VLDim object is a collection of vague points and vague lines. To represent a soft classification of space we propose a type VPartition. Figure 3 shows the vague spatial types and their relations, subclass and aggregation. The type SVSpatial represents an object of any of the simple types VPoint, VLine, or VRegion. Type GVSpatial represents an object of any of the general types VMPoint, VMLine, or VMRegion. VSpatial represents a (vague) spatial object of any type.

A simple or a general type represents an object whose essential property is expressed in vague terms. Such an object cannot be characterized only by the set of locations that form its extent. Any location is associated with a degree of membership to the object extent. An object is thus characterized by a function that determines the membership degree at each location. We expect vague objects to have mostly gradual transitions of membership values. Step-wise jumps of membership values may also occur. Membership values range between 0 and 1, normally covering the whole range [0, 1]. There are however applications that might need a finite set of memberships. It is desirable to define the vague types such that they include crisp objects as special cases. Semi-continuous functions satisfy all the above, being mostly continuous functions that allow jumps. All simple and general types are defined as fuzzy sets in $\mathbb{R}^2$ that satisfy some well-defined properties. To express these properties we use the fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ of semi-continuous functions in $\mathbb{R}$ and $\mathbb{R}^2$, respectively.

The section is organized as follows. Section 5.1 provides definitions of types and operators for vague point objects. Section 5.2 provides for types and operators of vague line objects. Section 5.3 provides types and operators for vague region objects. Vague partitions and their operators are discussed in section 5.4. For the

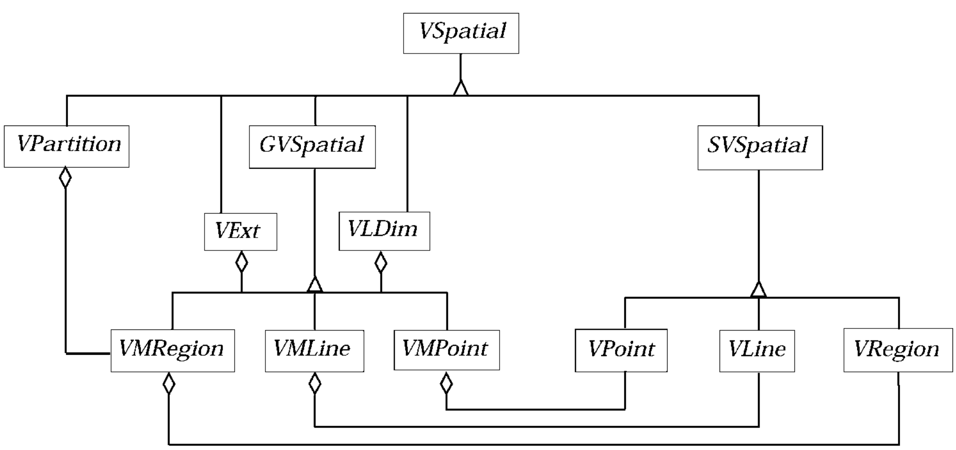

Figure 3. The hierarchy of vague spatial types.

illustrations of vague objects we use grey levels to show membership values. Dark tones indicate high membership values, light tones indicate low memberships.

5.1 *Vague point types and operators*

We propose two types of vague point objects: a *vague point* representing the simplest identifiable point object, and *a vague multipoint* representing a class of simple point objects. Figures 4(*a*) and (*b*) illustrate a vague point and a vague multipoint, respectively.

A *vague point* is a site with a known location, but with uncertain membership to a phenomenon of interest. It is defined as a fuzzy point p_λ in $\mathcal{T}_2$. The membership value λ represents the degree of belonging of the site $p(x, y)$ to the phenomenon of interest. The set of vague points is

$$\text{VPoint} \equiv \left\{\mu \in \mathcal{F}\left(\mathbb{R}^2\right) \middle| \exists!(x, y) \in \mathbb{R}^2,\ \mu(x, y) > 0\right\}.$$

The restriction of a fuzzy point μ to its support set is the singleton set $\{((x, y), \lambda)\}$. If the membership value $\mu(x, y)$ is equal to 1, then μ is a crisp point.

A *vague multipoint* is a finite collection of disjoint vague points. It is defined as a fuzzy set in $\mathbb{R}^2$ that has positive membership values in a finite set of locations. The set of vague multipoints is

$$\text{VMPoint} \equiv \left\{\mu \in \mathcal{F}\left(\mathbb{R}^2\right) \middle| \exists\{\mu_i\}_{i=1}^{n} \subset \text{VPoint},\ \mu = \bigsqcup_{i=1}^{n} \mu_i\right\}.$$

A vague point is a special case of a vague multipoint, for $n=1$. We allow the empty set $0^{\mathbb{R}^2}$ to be a special case of a vague multipoint, having $n=0$. This property is needed for the operators and VLDim type definition. The restriction of a vague multipoint to its support set is a finite set of triples providing vague point locations and their membership values: $\{((x_1, y_1), \lambda_1), ((x_2, y_2), \lambda_2) \ldots, ((x_n, y_n), \lambda_n)\}$.

The operators union, intersection and difference, for vague multipoints are the fuzzy operators, union $\sqcup$, intersection $\sqcap$ and difference $-$, respectively. The *union* between two vague multipoints is a vague multipoint of which the locations are the union of input point locations. The membership value at each location of the result is the maximum membership of input points at a common location, and simply the membership of the input point at any other location. Figure 5(*c*) illustrates the

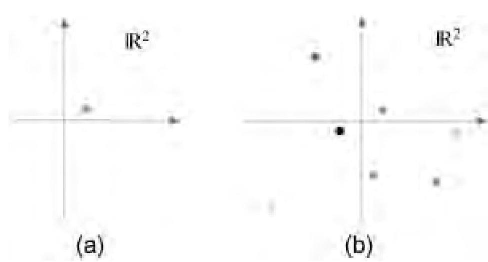

Figure 4. Vague point objects: (*a*) a vague point and (*b*) a vague multipoint.

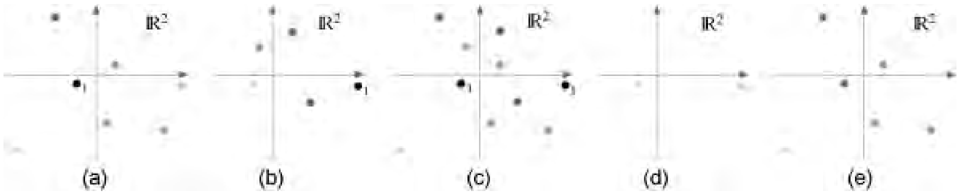

Figure 5. Results of vague multipoint operators: (*a*) and (*b*) two vague multipoints, (*c*)–(*e*) results of their union, intersection, and difference, respectively. Vague points with membership value equal to 1 are labelled with the value 1.

union of two vague multipoints of figures 5(*a*) and (*b*). The operator PUnion is defined as

$$\text{PUnion} : \text{VMPoint} \times \text{VMPoint} \rightarrow \text{VMPoint}$$
$$\forall \mu, \nu \in \text{VMPoint}, \text{PUnion}(\mu, \nu) = \mu \sqcup \nu.$$

The *intersection* between two vague multipoints is a vague multipoint of which the locations are the common locations of input points. The membership value at each location of the result is the minimum of memberships of input points at that location. Figure 5(*d*) illustrates the intersection of the vague multipoints of figures 5(*a*) and (*b*). The operator PIntersection is defined as

$$\text{PIntersection} : \text{VMPoint} \times \text{VMPoint} \rightarrow \text{VMPoint}$$
$$\forall \mu, \nu \in \text{VMPoint}, \text{PIntersection}(\mu, \nu) = \mu \sqcap \nu.$$

The *difference* of two vague multipoints is a vague multipoint of which the locations are those of the first input object. The membership value at a common location is the minimum of the membership of the first object and the complemented membership of the second object. At all other locations it is the membership of the first object. Figure 5(*e*) illustrates the difference of the vague multipoint of figure 5(*a*) with the vague multipoint of figure 5(*b*). The operator PDifference is defined as

$$\text{PDifference} : \text{VMPoint} \times \text{VMPoint} \rightarrow \text{VMPoint}$$
$$\forall \mu, \nu \in \text{VMPoint}, \text{PDifference}(\mu, \nu) = \mu - \nu.$$

The set of vague multipoints is closed under these operators, i.e. the union, intersection, and difference of two VMPoint objects is a VMPoint object. It can be seen that PUnion, PIntersection and PDifference applied to crisp multipoints are equivalent to the point set union, intersection and difference, respectively. Hence, they give the corresponding crisp operator when applied to crisp multipoints.

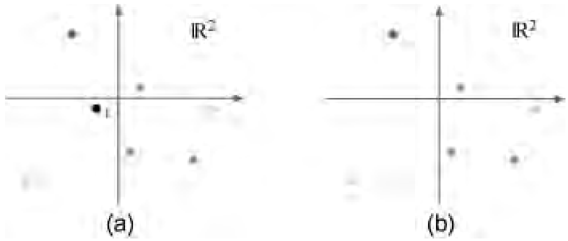

Figure 6. Boundary of a vague multipoint: (*a*) a vague multipoint and (*b*) its boundary.

The *boundary* of a vague multipoint μ is its uncertain part, i.e. the locations with membership value smaller than 1. It is constructed from the fuzzy boundary μ^{b} for the relative topology $\mathcal{T}_2^{\mu}$. The operator PBoundary is defined as

$$\mathrm{PBoundary} : \mathrm{VMPoint} \rightarrow \mathrm{VMPoint}$$

$$\forall \mu \in \mathrm{VMPoint},\ \mathrm{PBoundary}(\mu) = \mu^{\mathrm{b}}.$$

The boundary of a vague multipoint μ is its restriction to locations with positive membership smaller than 1: $\mathrm{PBoundary}(\mu) = \mu_{|\{(x,y) \in \mathbb{R}^2 | 0 < \mu(x,y) < 1\}}$. Figure 6(*b*) illustrates the boundary of the vague multipoint of figure 6(*a*).

The *frontier* of vague multipoint μ is empty. It is constructed from the fuzzy frontier μ^{f} for the relative topology $\mathcal{T}_2^{\mu}$. The frontier PFrontier of a vague multipoint is:

$$\mathrm{PFrontier} : \mathrm{VMPoint} \rightarrow \mathrm{VMPoint}$$

$$\forall \mu \in \mathrm{VMPoint},\ \mathrm{PFrontier}(\mu) = \mu^{\mathbb{R}^2}.$$

The boundary and the frontier of a crisp multipoint are empty. Thus, both operators give the crisp boundary operator when applied to crisp multipoints.

5.2 *Vague line types and operators*

We propose two types for representing vague objects of a linear nature. A *vague line* represents a linear object of the simplest structure, and a *vague multiline* represents a collection of vague lines that have the same membership value at their intersection points. Figures 7(*a*) and (*b*) illustrate a vague line and vague multiline, respectively. Section 5.2.1 provides description and definitions of vague line types, and section 5.2.2 provides for vague line operators.

Figure 7. Vague line objects: (*a*) a vague line and (*b*) a vague multiline.

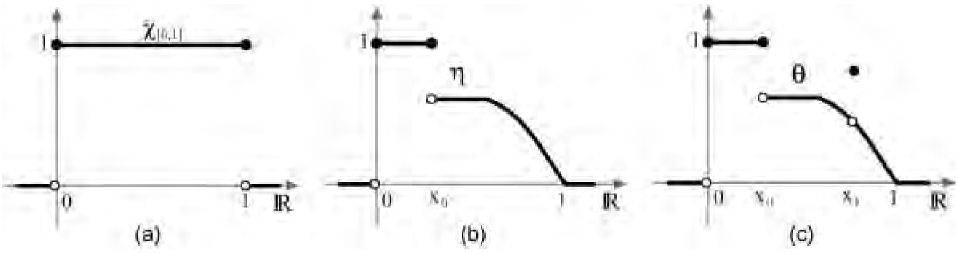

Figure 8. Membership functions for vague lines: (*a*) the characteristic function $\chi_{[0,\ 1]}$ of the closed interval [0, 1], (*b*) a function η having stepwise discontinuity at x_0, and (*c*) a function θ having an isolated discontinuity at x_1—this is not a valid membership function.

5.2.1 Vague line types. A *vague line* is a linear feature with known position, but with an uncertain extent, i.e. any point of the line has some certainty degree of belonging to the line. A vague line is a simple curve with mostly gradual transitions of membership values between neighbour points on the line. Membership values are positive at every location on the line, except, perhaps, at the end nodes. Stepwise changes of membership values may occur along the line, but isolated discontinuities are not permitted. The functions $\chi_{[0,\ 1]}$ and η of figures 8(*a*) and (*b*) are mostly continuous, both having stepwise changes at points 0 and 1, and at points 0 and x_0, respectively. Both functions have the type of continuity we want for membership functions along vague lines. The function of figure 8(*c*) has an isolated discontinuity at x_1, and is therefore not a valid membership function for a vague line.

We want the extension of a vague line to be a simple curve, which means it is a continuous, non-self intersecting curve, but possibly looped. (We call this a crisp line from here onwards.) The membership values of the vague line are given by a membership function defined over its extent. We require this function to be almost everywhere continuous, allowing stepwise changes in a finite number of locations.

A crisp line is topologically equivalent to the unit interval [0, 1], i.e. there is a homeomorphism h from [0, 1] to the line in $\mathbb{R}^2$. We can build a fuzzy set in [0, 1] that satisfies the continuity properties for membership functions, then transfer its membership values to the crisp line via the homeomorphism h. Figure 9(*b*) illustrates the construction of a vague line μ from the fuzzy set η of figure 8(*b*), via the homeomorphism h. The set η is drawn in figure 9(*b*) using saturation for displaying

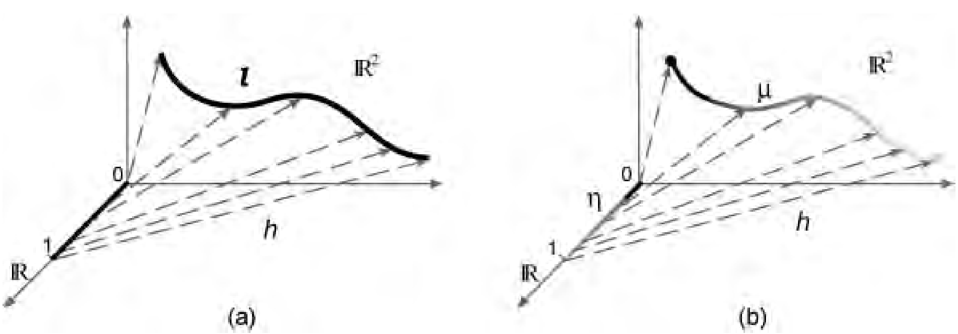

Figure 9. Construction of a crisp and vague line from sets in $\mathbb{R}$: (*a*) crisp line built from the homeomorphism h in (0, 1) and (*b*) vague line built by transferring the membership values of a fuzzy set η via the homeomorphism h.

membership values. The vague line μ is built from the extension principle as the image $\widetilde{h}(\eta)$ of the fuzzy set $\eta \in \mathbb{R}$.

A crisp line is the image of the [0, 1] interval by the homeomorphism h: $\{h(t)=(x(t), y(t)) | t \in [0, 1]\}$. To allow looped lines, the homeomorphism is restricted to (0, 1), requiring continuity at the end points 0 and 1 (Dilo 2000). An upper semi-continuous function satisfies most of the properties we want for the membership function of a vague line, but allows isolated discontinuities, e.g. the function of figure 8(c). The regular closure removes such isolated discontinuities. A vague line can now be built from a regular closed fuzzy set η in [0, 1] and the homeomorphism h. To assure that the vague line has a continuous extent, we require the fuzzy set η to have positive membership in [0, 1], that is η is connected for $(\mathbb{R}, \mathcal{T}_1)$. A vague line is thus built as the image by a homeomorphism of a regular closed and connected fuzzy set in [0, 1]. When the vague line is looped, the membership values at both end nodes are equal. The set of vague lines is defined as

$$\begin{aligned} \mathrm{VLine} \equiv \Big\{ \mu \in \mathcal{F}\big(\mathbb{R}^2\big) \,|\, \exists \eta \in \mathcal{F}([0, 1]),\ \eta = \overline{\eta^{\circ}}, \text{ and connected}, \\ \exists h : [0, 1] \to \mathbb{R}^2 \text{ homeomorphism in } (0, 1), \text{ continuous in } \{0, 1\}, \\ \mu = \widetilde{h}(\eta) \text{ and } (h(0) = h(1) \Rightarrow \eta(0) = \eta(1)) \Big\}. \end{aligned}$$

The homeomorphism h builds the extension of a vague line μ as topologically equivalent with the interval (0, 1) that is a 1-dimensional set. The extension of a vague line cannot be a finite set of points. Hence, the type vague line is different from the vague point and vague multipoint types. If the fuzzy set η in [0, 1] is a crisp set, then the vague line is a crisp line.

A *vague multiline* is a finite collection of vague lines, of which the extensions intersect only at their end nodes, and the lines have the same membership value at the common end nodes (see figure 7(b)). It is constructed from the union of vague lines from the finite collection. The set of vague multilines is

$$\mathrm{VMLine} \equiv \big\{ \mu \in \mathcal{F}\big(\mathbb{R}^2\big) \,|\, \exists \{\mu_i\}_1^n \subset \mathrm{VLine},\ \mu = \sqcup_{i=1}^n \mu_i \big\}.$$

A vague line is a special case of a vague multiline, having $n=1$. If $n=0$ the vague multiline is the empty set, a property required for the definitions of VLDim and VExt types.

A union type VLDim has values that are collections of vague lines and vague points. The type is defined as

$$\big\{ \mu \in \mathcal{F}\big(\mathrm{R}^2\big) \,|\, \exists \nu \in \mathrm{VMLine},\ \exists \gamma \in \mathrm{VMPoint},\ \mu = \nu \sqcup \gamma \big\}.$$

A VLDim object μ can be a vague multiline if the point component γ is empty, and it can be a vague multipoint if its line component ν is empty.

5.2.2 Vague line operators. The union, intersection and difference operators for vague lines are built from the corresponding fuzzy set operators. The *union* of two vague multilines is a vague multiline produced by the fuzzy set union of the input line objects. The union operator LUnion is defined as

$$\mathrm{LUnion} : \mathrm{VMLine} \times \mathrm{VMLine} \to \mathrm{VMLine}$$

$$\forall \mu, \nu \in \mathrm{VMLine},\ \mathrm{LUnion}(\mu, \nu) = \mu \sqcup \nu.$$

The set of vague multilines is closed under union.

The *intersection* of two vague multilines produces the intersection points of the two line extensions, associated by the membership values at these points calculated from the fuzzy intersection operator. The (point) intersection of the extensions of two vague multilines is produced by the intersection operator for crisp lines. Let μ and v be two vague multilines. We denote by EI the (crisp) intersection of their extensions: EI(μ, v)=Intersection(*supp*(μ), *supp*(v)). The intersection operator between vague multilines is defined from the fuzzy restriction to the intersection of their extensions:

$$\text{LIntersection} : \text{VMLine} \times \text{VMLine} \rightarrow \text{VMPoint}$$

$$\forall \mu,\ v \in \text{VMLine},\ \text{LIntersection}(\mu,\ v) = (\mu \sqcap v)_{|\text{EI}(\mu,\ v)}.$$

The difference operator between two vague multilines produces a multiline taken from the fuzzy difference of the fuzzy sets. The extension of the result is the (classical set) difference of the extensions of the input vague multilines. Membership values along the extension are calculated from the fuzzy difference. If two vague multilines μ and v intersect at points, there might be isolated discontinuity at the result of the fuzzy difference. To correct for this, we take the regular closure of fuzzy difference $\overline{(\mu - v)^{\circ}}$ (in the relative topology $\mathcal{T}_2^{\mu - v}$). The difference operator is then defined as

$$\text{LDifference} : \text{VMLine} \times \text{VMLine} \rightarrow \text{VMLine}$$

$$\forall \mu,\ v \in \text{VMLine},\ \text{LDifference}(\mu,\ v) = \overline{(\mu - v)^{\circ}}.$$

These three operators give the corresponding crisp operators when applied to crisp multilines.

The *boundary* of a vague multiline is its uncertain part. It is constructed from the union of boundaries of its vague line components. For a vague line μ expressed by a fuzzy set η and a homeomorphism h, the boundary is constructed as the image by h of the fuzzy boundary of η, $\widetilde{h}(\eta^{\mathrm{b}})$. When the vague line μ is crisp, its boundary consists of vague points that are the end nodes of the line. When the membership function along the line is continuous, which means that η is continuous, the boundary of μ consists of vague line components. In general, the boundary of a vague line consists of vague points and vague lines. The boundary of a vague multiline $\mu = \sqcup_{i=1}^{n} \mu_i$ is the union of the boundaries of its components μ_i. The boundary operator for vague multilines is defined as

$$\text{LBoundary} : \text{VMLine} \rightarrow \text{VLDim}$$

$$\forall \mu \in \text{VMLine},\ \mu = \sqcup \left\{ \mu_i \middle| \mu_i = \widetilde{h}_i(\eta_i) \in \text{VLine},\ i \in \{1 \ldots n\} \right\},$$

$$\text{LBoundary}(\mu) = \sqcup_{i=1}^{n} \widetilde{h}_i\left(\eta_i^{\mathrm{b}}\right).$$

Figure 10(a) illustrates a vague multiline. The end nodes of its core are shown by empty circles. Figure 10(b) shows its boundary, which consists of vague lines and vague points. The boundary of a vague multiline μ is the restriction to the set of locations with positive membership smaller than 1, extended by the boundary of the core.

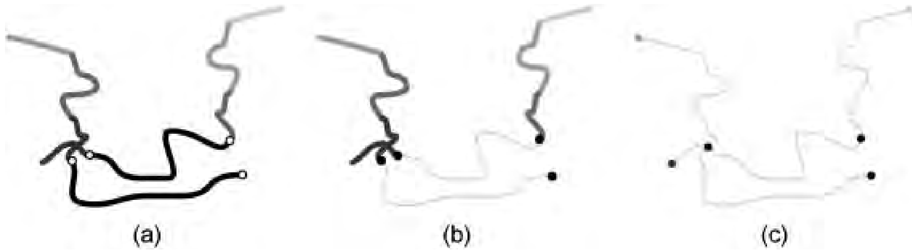

Figure 10. Boundary and frontier of a vague multiline: (*a*) a vague multiline with its core boundary shown in empty circles, (*b*) its boundary (the core is drawn in light grey) and (*c*) its frontier (the line extension is drawn in light grey).

The *frontier* of a vague multiline is constructed in a similar way from the frontiers of its vague line components. The frontier of a vague line $\mu = \widetilde{h}(\eta)$ is the image of the fuzzy frontier of η, $\widetilde{h}(\eta^{\mathrm{f}})$. The frontier of a vague multiline is the union of frontiers of its vague line components, and is a vague multipoint. The operator is defined as

$$\mathrm{LFrontier} : \mathrm{VMLine} \rightarrow \mathrm{VMPoint}$$

$$\forall \mu \in \mathrm{VMLine},\ \mu = \sqcup \left\{ \mu_i \middle| \mu_i = \widetilde{h}_i(\eta_i) \in \mathrm{VLine},\ i \in \{1 \ldots n\} \right\},$$

$$\mathrm{LFrontier}(\mu) = \sqcup_{i=1}^{n} \widetilde{h}_i\left(\eta_i^{\mathrm{f}}\right).$$

Figure 10(*c*) shows the frontier of the vague multiline of figure 10(*a*). The frontier of a vague multiline is its restriction to the set of discontinuity locations of the membership function.

The boundary LBoundary and the frontier LFrontier applied on a crisp multiline produce the set of end nodes of its line components. Thus, both operators give the crisp boundary operator when applied to crisp multilines.

5.3 *Vague region types and operators*

We distinguish two types of vague region objects, *vague region* representing the simplest identifiable object, and *vague multiregion* representing a class of vague regions. A vague region is a single-component fuzzy set that does not have irregularities: isolated vague points and vague lines, or punctures and cuts, i.e. removed vague points and vague lines, respectively. A vague multiregion is a collection of disjoint vague regions. The fuzzy set of figure 11(*a*) has a puncture and a cut, both irregularities that are not allowed for a vague region object. Figure 11(*b*) illustrates a vague region, and figure 11(*c*) illustrates a vague multiregion.

Section 5.3.1 provides definitions for vague regions, vague multiregions and the type vague extent. Section 5.3.2 provides definitions for vague region operators. Illustrations for both sections are produced from data on heavy metal concentration in the sediments of the Maas river.

5.3.1 Vague region types. A *vague region* is a broad boundary region, such that points in the broad boundary typically have different membership values, which change mostly gradually between neighbour points. The membership values can change abruptly along a line, making a stepwise jump. Abrupt changes only at one location, or membership values along a line changing abruptly from both sides, are

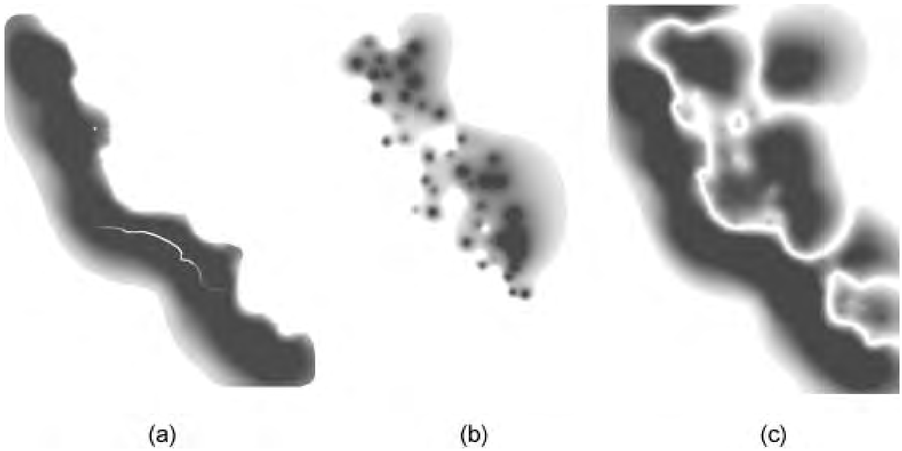

Figure 11. Fuzzy sets in $\mathbb{R}^2$: (*a*) a fuzzy set that is not a vague region, (*b*) a vague region and (*c*) a vague multiregion.

not allowed. Figure 12 illustrates different vague regions. The vague region of figure 12(*a*) has a single-component core and does not have holes. Its membership function decreases gradually from the boundary of the core to the boundary of the support set. Every α-cut of the region is connected. The vague region of figure 12(*b*) has two cores. The vague region of figure 12(*c*) has a single-component core, and it contains holes. All its α-cuts are connected. The vague region of figure 12(*d*) has several cores and several holes.

We want the support set of a vague region to be a crisp region, and the membership function to be almost everywhere continuous in the support set, allowing stepwise jumps along linear features. We require a vague region to be bounded, regular closed, and with connected interior in $(\mathbb{R}^2, \mathcal{T}_2)$. When a fuzzy set satisfies these three properties for $\mathcal{T}_2$, its support set satisfies the same properties for T_2, which means it is a crisp region. The regular closure assures stronger properties than the upper semi-continuity: the discontinuities are stepwise jumps along lines; no isolated discontinuities are allowed, and no discontinuity from both sides of a line occur. It satisfies the membership function requirement. The set of vague regions is then defined as

$$\text{VRegion} \equiv \left\{ \mu \in \mathcal{F}\left(\mathbb{R}^2\right) | \mu \text{ is bounded}, \; \mu = \overline{\mu^\circ}, \; \mu^\circ \text{ is connected} \right\}.$$

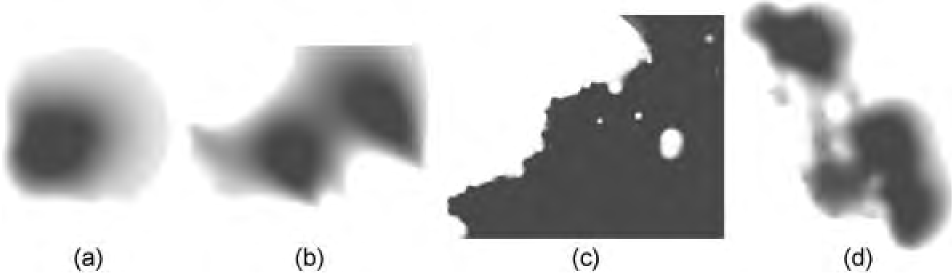

Figure 12. Vague region objects.

The highest membership value may be less than 1. The regular closure property for $\mathcal{T}_2$ excludes the possibility for a vague line or vague multiline to be a vague region. A crisp region is a specific case of a vague region, when the set μ is a crisp set.

A *vague multiregion* represents a class of vague region objects. It is a multi-component fuzzy set that is bounded and regular closed. Figure 11(*b*) illustrates a vague multiregion with only one component, and figure 11(*c*) illustrates a multi-component region. The set of vague multiregions is defined as

$$\mathrm{VMRegion} \equiv \left\{\mu \in \mathcal{F}\left(\mathbb{R}^2\right) | \mu \text{ is bounded, } \mu = \overline{\mu^\circ}\right\}.$$

A vague region is a special case of a vague multiregion, being a region with a single component. A vague multiregion can also be empty.

A vague extension is a collection of vague multiregions and vague multilines. The set of vague extensions is defined as

$$\mathrm{VExt} \equiv \left\{\mu \in \mathcal{F}\left(\mathbb{R}^2\right) | \exists \nu \in \mathrm{VMRegion},\ \exists \gamma \in \mathrm{VMLine},\ \mu = \nu \sqcup \gamma\right\}.$$

A VExt object μ can be a vague multiregion if the line component γ is empty, and it can be a vague multiline if its region component ν is empty.

5.3.2 Vague region operators. The union, intersection and difference operators for vague regions are regularized fuzzy set operators. The type VMRegion is closed under these operators, i.e. the union, intersection or difference of two vague multiregions is a vague multiregion. Figure 13(*a*) and (*b*) show two vague multi regions that are overlaid using transparency in Figure 13(*c*). The results of union, intersection, and difference between these two regions are shown in Figure 14(*a*)–(*c*).

The *union* of two vague multiregions is simply the fuzzy set union. The union of two bounded fuzzy sets is a bounded set. The union of two regular closed fuzzy sets is a regular closed fuzzy set. Therefore, the fuzzy set union of two vague multiregions produces a vague multiregion. The union operator RUnion is defined as

$$\mathrm{RUnion} : \mathrm{VMRegion} \times \mathrm{VMRegion} \rightarrow \mathrm{VMRegion}$$

$$\forall \mu, \nu \in \mathrm{VMRegion},\ \mathrm{RUnion}(\mu, \nu) = \mu \sqcup \nu.$$

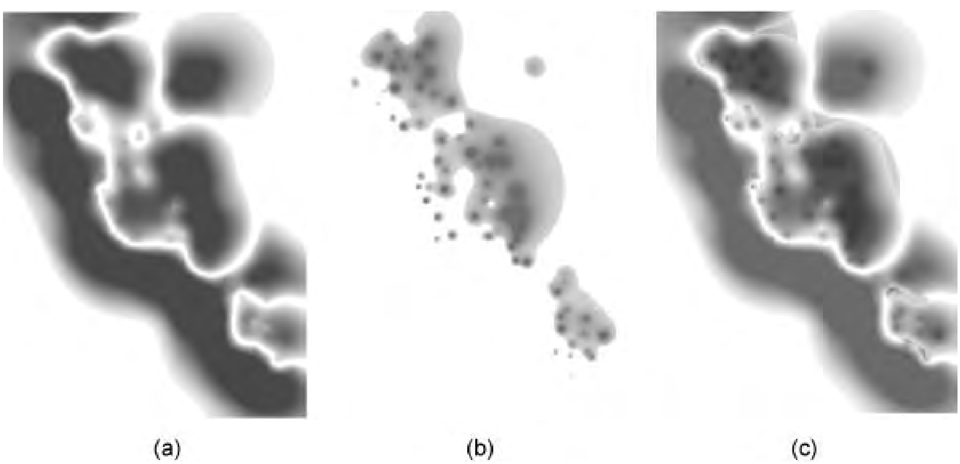

Figure 13. Two vague multiregions overlayed: (*a*) and (*b*) vague multiregions, (*c*) overlayed and displayed by using transparency for the top region.

Figure 14. Results of operators on vague multiregions of figure 13: (*a*)–(*c*) union, intersection and difference, respectively.

The *intersection* of two vague multiregions is the regular closure of their fuzzy set intersection. Fuzzy intersection of two bounded fuzzy sets is a bounded fuzzy set. Fuzzy intersection of two regular closed fuzzy sets is not always regular closed. We obtain a vague multiregion by applying the regular closure on the result of the fuzzy intersection of two vague multiregions. The interior of a fuzzy intersection is equal to the fuzzy intersection of the interiors. Therefore, we can define the intersection operator between regions as

$$\text{RInterection} : \text{VMRegion} \times \text{VMRegion} \rightarrow \text{VMRegion}$$
$$\forall \mu,\ \nu \in \text{VMRegion},\ \text{RInterection}(\mu, \nu) = \overline{\mu^{\circ} \sqcap \nu^{\circ}}.$$

The *difference* operator between two vague multiregions μ and ν is built from the fuzzy difference, which in turn is defined in terms of a fuzzy intersection: $\mu \sqcap \left(1_{\mathbb{R}^2} - \nu\right)$. The fuzzy difference between two vague multiregions produces a bounded fuzzy set, but not always a regular closed fuzzy set. Again, we apply the regular closure on the result of the fuzzy operator, in order to get a vague multiregion. The interior of the fuzzy intersection is equal to the intersection of the interiors, and the complement of ν is an open set. We can, therefore, define the difference between vague multiregions as

$$\text{RDifference} : \text{VMRegion} \times \text{VMRegion} \rightarrow \text{VMRegion}$$
$$\forall \mu,\ \nu \in \text{VMRegion},\ \text{RDifference}(\mu, \nu) = \overline{\mu^{\circ} \sqcap \left(1_{\mathbb{R}^2} - \nu\right)}.$$

The *boundary* of a vague multiregion μ is its uncertain part, and it is constructed from the fuzzy boundary μ^{b}. The boundary of a vague multiregion may consist of vague regions, vague lines, or both. It is a vague extension VExt. The boundary of a crisp region consists only of lines, whereas the boundary of a vague multiregion with continuous membership function is a vague multiregion. Figure 15(*a*) illustrates a vague multiregion with its core boundary drawn in grey, and figure 15(*b*) shows its

Figure 15. Boundary and frontier of a vague multiregion: (*a*) a vague region with the core boundary in grey, (*b*) its boundary and (*c*) its frontier.

boundary, which is a vague extension. The boundary operator is defined as:

$$\text{RBoundary} : \text{VMRegion} \rightarrow \text{VExt}$$

$$\forall \mu \in \text{VMRegion}, \ \text{RBoundary}(\mu) = \mu^{\text{b}}.$$

The boundary of a vague multiregion μ is the restriction of μ to locations with membership values smaller than 1, extended by the boundary of the core:

$$\text{RBoundary}(\mu) = \mu_{|\{p \in \mathbb{R}^2 | 0 < \mu(p) < 1\} \cup \partial \mu_1}.$$

The *frontier* of a vague multiregion μ is calculated as the fuzzy frontier μ^{f}. The frontier operator on vague multiregions returns a vague multiline. If μ has discontinuities, the frontier returns all lines of discontinuity. When μ is continuous, its frontier is empty. Figure 15(*c*) illustrates the frontier of the vague multiregion of figure 15(*a*). The operator RFrontier is defined as:

$$\text{RFrontier} : \text{VMRegion} \rightarrow \text{VMLine}$$

$$\forall \mu \in \text{VMRegion}, \ \text{RFrontier}(\mu) = \mu^{\text{f}}.$$

Both operators, the boundary and the frontier, produce the crisp boundary when applied to crisp multiregions.

5.4 *Vague partitions and their operators*

In practical applications, vague multiregions often originate from a soft classification of space, for example based on remote sensing imagery. Vague multiregions representing different classes may not be disjoint, as we expect transition zones to intersect with each other. A soft classification cannot give a crisp partition of space, but some characteristics of such a partition should be kept to make a meaningful classification. A *vague partition* serves this purpose. It is a collection of vague multiregions that may intersect only at their uncertain parts. The core of one region can intersect with the support set of the other region only at their boundaries. The set of vague partitions is defined as

$$\mathrm{VPartition} \equiv \left\{ \{\mu_i\}_{i=1}^{n} \subset \mathrm{VMRegion} | \forall p, \exists i \in \{1 \ldots n\}, \mu_i(p) > 0, \right.$$
$$\left. \forall_i, j, i \neq j \Rightarrow \mu_i \sqcap \mu_j \sqsubseteq \mathrm{RBoundary}(\mu_i) \sqcap \mathrm{RBoundary}(\mu_j) \right\}.$$

The first condition assures that every location has a positive membership to at least one vague class, and the second condition assures that any two classes may intersect only at their uncertain part.

The operators we define for vague partitions are the overlay and the fusion operator. The *overlay* operator VPOverlay superimposes two vague partitions, and creates a new vague partition with vague multiregions obtained from the intersections of a vague multiregion of the first partition with a vague multiregion of the second partition. It is defined as

$$\mathrm{VOverlay} : \mathrm{VPartition} \times \mathrm{VPartition} \rightarrow \mathrm{VPartition}$$
$$\forall \mathcal{P}_1 = \{\mu_i\}_{i=1}^{n}, \mathcal{P}_2 = \{v_j\}_{j=1}^{m} \in \mathrm{VPartition},$$
$$\mathrm{VOverlay}(\mathcal{P}_1, \mathcal{P}_2) = \left\{ \zeta_{i,j} \middle| i \in \{1 \ldots n\}, j \in \{1 \ldots m\}, \zeta_{i,j} = \mathrm{RInterection}(\mu_i, v_j) \right\}.$$

It can be shown that the set $\{\zeta_{i,j}|i \in \{1 \ldots n\}, j \in \{1 \ldots m\}\}$ forms a vague partition. The overlay operator combines two vague classifications of space, and creates a new classification that is more refined.

The *fusion* operator dissolves a vague partition by merging vague multiregions based on grouping or equality of some attribute value of regions. The operator assumes an attribute to be associated to vague multiregions of a vague partition. Let us call such a partition an *attribute extended vague partition*, and let us denote by ADomain the domain of attribute values. The set of such partitions is

$$\mathrm{AVPartition} \equiv \left\{ \{(\mu_i, v_i)\}_{i=1}^{n} \subset \mathrm{VMRegion} \times \mathrm{ADomain} \middle| \{\mu_i\}_{i=1}^{n} \in \mathrm{VPartition} \right\}.$$

For simplicity we consider one attribute attached to the vague multiregions of a partition. The set ADomain can generally be a Cartesian product of domains of several attributes. A grouping of attribute values is a function g:ADomain $\rightarrow$ADomain. This function is defined on the assumption that the group values are in the same domain ADomain. Such a function g is an element of the power set $\mathbb{P}$ (ADomain $\times$ ADomain), the collection of subsets of the Cartesian product of ADomain with itself. The fusion operator is then defined as

$$\mathrm{VFusion} : \mathrm{AVPartition} \times \mathbb{P}(\mathrm{ADomain} \times \mathrm{ADomain}) \rightarrow \mathrm{AVPartition}$$
$$\forall \mathcal{A} = \{(\mu_i, v_i)\}_{i=1}^{n} \in \mathrm{AVPartition}, \forall g \in \mathbb{P}(\mathrm{ADomain} \times \mathrm{ADomain}),$$
$$\mathrm{VFusion}(\mathcal{A}) = \left\{ \{(\zeta_j, w_j)\}_{j=1}^{m} \middle| \{w_j\}_{j=1}^{m} = \mathrm{ran}(g), \right.$$
$$\left. \forall j \in \{1 \ldots m\}, \zeta_j = \sqcup \left\{ \mu_i \middle| g(v_i) = w_j \right\} \right\}.$$

The generalized fuzzy union of vague multiregions has a vague multiregion as its output, therefore ζ_j's are VMRegion objects. It can be shown that the set $\{\zeta_j\}_1^m$ forms a vague partition. The fusion operator allows one to generalize a vague partition.

6. Other spatial operators

Abstract models for spatial data propose a list of types and fundamental operators that are needed in spatial data systems. The ROSE algebra (Güting 1994, Güting and Schneider 1995) and OpenGIS abstract specification (Open GIS Consortium 1999) both provide a similar list of spatial operators. (A comparison between the two models can be found in Dilo (2006).) To present the list of operators, we follow here the grouping provided in Güting (1994).

(i) Operators returning spatial data type values, e.g. *intersection* of regions—returning regions, *union* of lines—returning lines, *boundary* of regions—returning lines.
(ii) Spatial predicates expressing spatial relations, e.g. point *within* region, region *overlaps* region.
(iii) Spatial operators returning numbers. These are mainly metric operators e.g. *length* of a line, *area* of a region, *distance* between two regions.
(iv) Operators on collection of objects, e.g. *overlay* of partitions, *fusion* of regions based on the equality of values of a certain attribute.

Vague spatial types introduced in section 5 form a complete set of types describing spatial objects. Some numerical types are needed for the other operators, spatial predicates and metric operators. The type TruthDegree≡[0, 1] is the return type of all spatial predicates. Two other types, Measure$\equiv\mathbb{R}^+$, and VMeasure$\equiv\{f{:}(0,\ \sigma]\rightarrow\mathbb{R}^+|\sigma\in(0,\ 1]$ and f is semi-continuous$\}$ are needed for metric operators.

The vague operators introduced in section 5 cover the first and fourth group of spatial operators: operators returning spatial data types and operators on collections[1]. Other operators returning spatial types can be expressed as a combination of union, intersection and complement, e.g. difference and symmetric difference. In this section we give intuition and example definitions for operators of the two other groups, spatial predicates and metric operators.

Spatial predicates express relations between vague multipoints, vague multilines and vague multiregions. We provide relations Disjoint, Touches, Crosses, Overlaps, Within, and Equal, which is the set of relations proposed by SQL/MM spatial (ISO 1999). The relations extend the true/false set of truth values of the SQL/MM relations to the [0, 1] interval. That means that the truth of a relation is a matter of degree, thus a value of type TruthDegree. A value v between 0 and 1 for a relation R(μ, v) means that objects μ and v are in relation R to the degree v. A value 0 for R(μ, v) means that μ and v are certainly not in relation R, whereas a value 1 means the two are certainly in R. Spatial relations are defined from membership values of the objects involved, considering extreme values that support a relation or disapprove it. The complete treatment of spatial relations, intuition, definition and illustrations can be found in Dilo *et al.* (2005) and Dilo (2006). Here we give the properties of the relations, and some example definitions.

The relations Disjoint, Touches, Crosses and Overlaps between vague objects are defined such that a relation is certain if the corresponding crisp relation is true for their cores; a relation is certainly false if the corresponding

[1] More operators are proposed in the ROSE algebra from this last group.

crisp relation is false for their support sets. For example, the relation Disjoint is defined as:

$$\text{Disjoint} : \text{GVSpatial} \times \text{GVSpatial} \rightarrow \text{TruthDegree}$$
$$\forall \mu,\ v \in \text{GVSpatial},\ \text{Disjoint}(\mu,\ v) = 1 - \sup_{p \in \mathbb{R}^2}\{(\mu \sqcap v)(p)\}.$$

The total certainty of the other two relations, Within and Equal, is modelled by the subset and equality relation for fuzzy sets, respectively. A Within(μ, v) relation is certainly false if the corresponding crisp relation between the core of μ and the support set of v is false. Similarly, an Equal relation is certainly false if the corresponding crisp relation between the core of one object and the support set of the other is false. The relation Within is defined from the bounded difference between two fuzzy sets μ and v: $\forall p$, $\mu \nabla v(p) = \max\{0, \mu(p) - v(p)\}$. The Within relation is defined as:

$$\text{Within} : \text{GVSpatial} \times \text{GVSpatial} \rightarrow \text{TruthDegree}$$
$$\forall \mu,\ v \in \text{GVSpatial},\ \text{Within}(\mu,\ v) = \begin{cases} 0 & \text{if } \mu \sqcap v = 0^{\mathbb{R}^2}, \\ 1 - \sup_p\{(\mu \nabla v)(p)\} & \text{otherwise.} \end{cases}$$

The relations have the property that only one relation can be certain at a time, i.e. if one relation is certain, all the others have a degree smaller than 1. For some of the relations this property is stronger: if a relation is certain, all the others are false. Each relation gives the corresponding crisp relation when applied to crisp objects.

Metric operators that we provide are distance between two vague objects of any type, length of a vague multiline, area, diameter and perimeter of a vague multiregion. An operator on vague objects is such that for every α in (0, 1] it returns the value of the analogous crisp operator applied to the α-cuts of the vague objects. For example, the area operator returns for every α in (0, 1] the area of the α-cut of the vague multiregion. We call these alpha operators. An alpha operator takes as argument one or two vague objects, and returns a function from an interval in (0, 1] to the non-negative real numbers $\mathbb{R}^+$. The returned function by an alpha operator is a value of type VMeasure. Again, we give here only example definitions. The complete treatment of metric operators can be found in Dilo (2006).

The alpha area of a vague multiregion μ is calculated from the areas of its α-cuts: $\text{AREA}(\mu_\alpha) = \int\int_{\mu_\alpha} \mathrm{d}x\, \mathrm{d}y$. If the maximum of μ is lower than 1, we consider the area Area(μ) to be 0 for all α values higher than the maximum. The Area operator is defined as:

$$\text{Area} : \text{VMRegion} \rightarrow \text{VMeasure}$$
$$\forall \mu \in \text{VMRegion},\ \text{Area}[\mu](\alpha) = \begin{cases} \text{Area}(\mu_\alpha) & 0 < \alpha \leq \max_p\{\mu(p)\}, \\ 0 & \max_p\{\mu(p)\} < \alpha \leq 1. \end{cases}$$

For each alpha operator we provide a corresponding operator that produces an average over all values of the return function of the alpha operator. We call the operators of this second group average operators. As the integration performs an averaging process on functions, we define an average operator as the integral over

[0, 1] of the return function of the corresponding alpha operator. An average operator returns a non-negative real number that is a value of type Measure. As an example operator from this group see the average area operator.

The alpha area of a vague multiregion is a non-increasing and upper semi-continuous function, thus the function Area(μ) is integrable. The average area of a region μ is calculated from the integral of Area(μ) over [0, 1]. The operator AvArea is defined as:

$$\text{AvArea} : \text{VMRegion} \rightarrow \text{Measure}$$

$$\forall \mu \in \text{VMRegion},\ \text{AvArea}(\mu) = \int_0^1 \text{Area}[\mu](\alpha)\,\mathrm{d}\alpha.$$

The average area calculated from the integral Area[μ](α)dα provides the volume under the μ function. Therefore, the average area is equal to

$$\text{AvArea}(\mu) = \int\int \mu(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

This provides a way to calculate the average area operator independently from the alpha area operator.

An alpha operator can be used to provide a measure for a given level of membership, whereas an average operator provides an overall measure for vague object(s), performing an averaging over all membership levels. An average operator gives its crisp analogue when applied to crisp objects.

7. Illustration of vague objects from coastal erosion

We present here an application from coastal erosion (van de Vlag *et al.* 2004) to illustrate some of the object types and operators that have been introduced before. The application has its study area in the island of Ameland, in the north of the Netherlands (see figure 16). Strong tidal currents cause the movement of sand around the island. Subsequent erosion and sedimentation in turn cause major changes along the shore. Figure 17 shows the digital elevation models (DEM) of the northern part of Ameland in the years 1989–1995, in a grey scale picture. The dark part is the sea, the greyish part is the beach, and the white part are the dunes. From figure 17 it can be seen that the beach is moving and shrinking towards a south-easterly direction. The Ministry of Public Works of The Netherlands is interested in

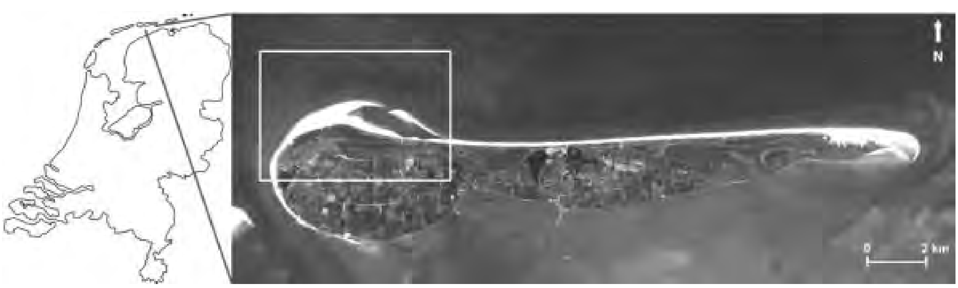

Figure 16. The study area: the north-west part of the Ameland island, north of The Netherlands (taken from van de Vlag (2006)).

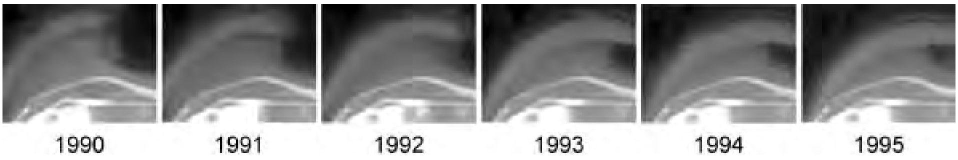

Figure 17. DEM of the northern part of Ameland for years 1990–1995. Low elevation values are shown in black colour, high elevation values in white. The dark part is the sea, the grey part is the beach and the white part are the dunes.

stabilizing the process of change in the Ameland shore. To neutralize the erosion they carry out beach nourishment by depositing sand. They are interested to know where the nourishment should be performed. Beach areas without vegetation have the highest erosion risk, hence they are the areas where sand has to be deposited.

Survey data from the JARKUS database (van de Vlag *et al.* 2004) and LandsatTM5 images are used for this application. The digital terrain model (DTM) is built from survey data, and the vegetation index (NDVI) is calculated from Landsat images. Beach areas are derived from the DTM, and the non-vegetated areas are derived from the NDVI index. In the methodology of the Dutch National Institute for Coastal and Marine Management (RIKZ) for the preservation of the coast, beach areas are considered those having elevations between -1 and $2\,\text{m}$ (a short description is provided in van de Vlag *et al.* (2004)). The light grey bars in figure 18 show the crisp classification of the zone performed by RIKZ. However, the separation between the three classes, foreshore, beach and foredune, is not crisp. Transition zones occur between these three classes, hence justifying an approach by vague objects. The concepts 'vegetated' and 'non-vegetated' are also vague. There are transition zones with scarce vegetation. An NDVI value around 0 separates vegetated from non-vegetated areas.

Emerging objects in this application are vague partitions and vague region objects. The vague regions are a typical example of regions resulting from a classification over space. The three vague classes, foreshore, beach and foredune, are vague multiregions. They are calculated from the elevation values applying the membership functions shown in figure 18. Results of the crisp and fuzzy classifications for the year 1995 are shown in figure 19. Part (*a*) is a map of elevations in the study area, part (*b*) shows the result of the crisp classification of elevations, and parts (*c*)–(*e*) are the vague classes created from the membership functions of figure 18. The three vague classes form a vague partition of the zone, assured from the choice of their membership functions.

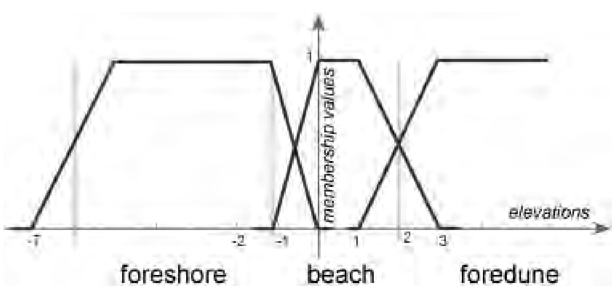

Figure 18. Membership functions for shore, beach and dune. The light grey bars show the boundaries for a crisp classification of elevations.

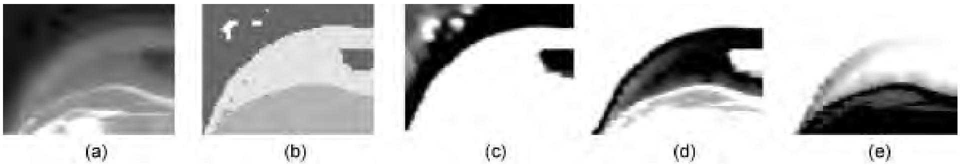

Figure 19. Elevation and results of crisp and fuzzy classification: (*a*) DEM of Ameland for the year 1995, (*b*) crisp classification of elevations, (*c*) vague shore, (*d*) vague beach, (*e*) vague dune. Saturated colour shows high membership for vague objects.

Figure 20(*a*) shows the NDVI index for year 1995. Figure 20(*b*) shows the membership functions applied on NDVI values to derive vegetated and non-vegetated classes. A transition zone exists when passing from vegetated to non-vegetated objects: NDVI values in $[-0.05, 0.05]$ are taken to have a positive degree of being both vegetated and non-vegetated at the same time (figures 20(*c*) and (*d*)). The lower part of the area is covered by dune vegetation, tall grass and occasional bush. This corresponds to high membership values of vegetated objects shown in figure 20(*d*). Further north starts the flat area with only some sparse vegetation. This causes an abrupt change of membership values inside the lower-most vegetated object. Both classes, vegetated and non-vegetated are vague multiregions, and together form a vague partition of the area.

Sand is to be deposited at non-vegetated beach areas that have a certain extent, i.e. the area of such an object is over a certain limit. Non-vegetated beach objects are derived from the intersection of beach and non-vegetated vague classes (figures 21(*a*) and (*b*)). Their intersection shown in figure 21(*c*) is again a vague multiregion object. It is composed of two vague regions. The average area AvArea operator is applied on vague regions to decide whether to perform nourishment or not. Vague regions with average area higher than a certain limit are selected for the nourishment. The lower region has a very small (average) area. Only the upper region will be selected for nourishment.

Illustrations and calculations for this section are done in ArcGIS, using raster data to present a vague class, and VisualBasic programming to perform the area calculation for a class. The average area can be easily calculated for the whole class (raster layer), whereas its calculation for a vague region object cannot be done. Selection of a region object, a component in the beach class, is not possible in ArcGIS raster data. In Dilo *et al.* (2006) we show a possible implementation for the vague spatial types. A vague region, stored in a TIN-like structure, can be selected in

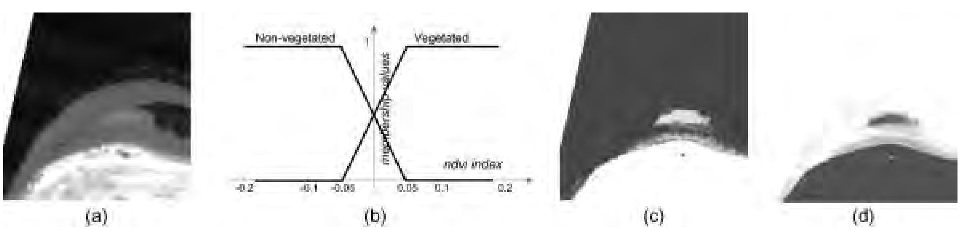

Figure 20. Vegetation index and fuzzy classification: (*a*) NDVI index of Ameland for the year 1995, low values in dark tone, high values in light tone, (*b*) membership functions applied to define non-vegetated and vegetated classes, (*c*) and (*d*) non-vegetated and vegetated vague objects.

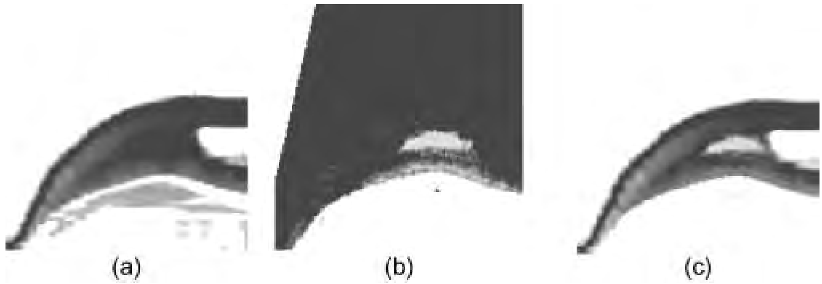

Figure 21. Vague multiregions: (*a*) beach, (*b*) non-vegetated and (*c*) beach plain.

the screen and its attribute values can be displayed. We did not implement the metric operators, therefore we could not use that system for the complete application.

8. Discussion and conclusions

This paper provided a system of types and operators for handling vague spatial objects. Research concentrates on objects in $\mathbb{R}^2$: vague point, line and region objects. The proposed types are formal representations of spatial objects that exhibit thematic vagueness. Locational vagueness is not always covered by the proposed types. Vague region types can often cover locational vagueness, whereas vague point and line types assume known (crisp) location. Regions often result from a classification of space. The vagueness of concepts defining the classes brings vagueness to the regions extent. This means that locational vagueness derives from thematic vagueness. Therefore, the proposed region types can represent this kind of locational vagueness as well.

To present points and lines exhibiting locational vagueness we could use the vague region types with an additional restriction that the membership value should not reach the value 1. This restriction is based on the assumption that a location with a membership value equal to 1 gives with certainty the point, or a part of the line extension. The real differentiation between the objects, points, lines and regions would then be left to their operators, which should hold a different semantic for each object type.

The proposed vague spatial types were grouped into simple types and general types. Simple types represent identifiable objects of a simple structure, whereas general types assure closure under operators. The membership functions of vague objects are mostly continuous, but can have stepwise jumps. Definitions of types allow for a finite or infinite set of membership values. Therefore, broad boundary regions can be represented by our types. Crisp types are also special cases of the vague types presented here.

The provided operators allow spatial analysis and reasoning over the vague spatial types. They were divided into three groups: operators returning spatial types, spatial relations and metric operators. In this paper we put more attention to the first group of operators, giving their description and definitions. Operators of this group consist of regularized fuzzy set operators, boundary operators, and operators on partitions. For the operators of the other two groups we gave the intuition and some example definitions. An operator on vague objects includes a crisp operator as a special case, which means that when applied to crisp objects it returns the same result as the corresponding crisp operator.

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